LECTURE #06 _ Topic 2 (Contd).

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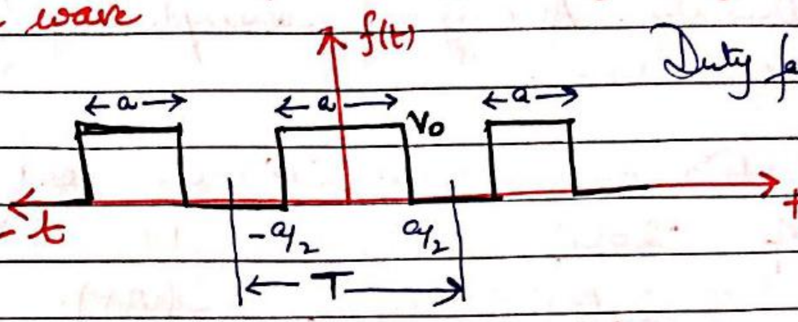
T Sheet # (02) Contd → Fourier Transform

Revised : Exclude Q.18; Q19(1,2,3); Q.23; Q.24

Q.17(a) Draw the Double sided Amplitude Spectrum and Corresponding Double sided Phase spectrum of the following periodic Square wave



Duty factor a/T

Attempt for a(i) $\frac{a}{T} = \frac{1}{2}$; a(ii) $\frac{a}{T} = \frac{1}{4}$; a(iii) $\frac{a}{T} = \frac{1}{5}$

Q.17(b) In general if $a/T = \frac{1}{N}$; then as N increases What will be the influence of a change in $\frac{a}{T}$ duty factor on the following

b(i) Spectrum

b(ii) Spacing $\Delta\omega$ between Components

b(iii) Peak amplitude of Spectrum

b(iv) First Zero Crossing

b(v) Bandwidth Consideration

Q.18 Excluded. and OMIT & do not attempt Q.18.

Q.19(1)(2)(3) Exclude and OMIT 19(1); 19(2); 19(3).

Q.19 Find the Fourier Transform of following functions and draw the corresponding Amplitude & Phase spectrum

Q 19(4) A Delta function ; $f(t) = \delta(t)$

Q 19(5) $f(t) = 1$

Q 19(6) $f(t) = \exp(j\omega_0 t)$

Q 19(7) $f(t) = \exp(-j\omega_0 t)$

Q.19(8) Define the Property of Reciprocal Spreading & Duality. Also give examples of its ~~applied~~ application.

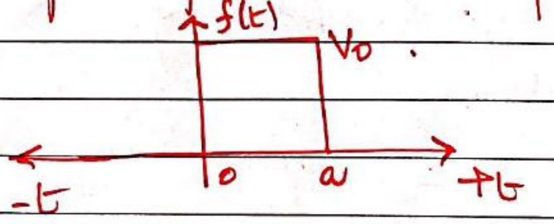
Q.20 (1,2) Obtain Fourier Transform ~~&~~ with neat illustration of

20(1) $f(t) = \cos(\omega_0 t)$

20(2) $f(t) = \sin(\omega_0 t)$.

Q.21. Determine the Fourier Transform of an impulse train

Q.22 Obtain Fourier Transform and plot Frequency & phase spectrum where pulse is shown below



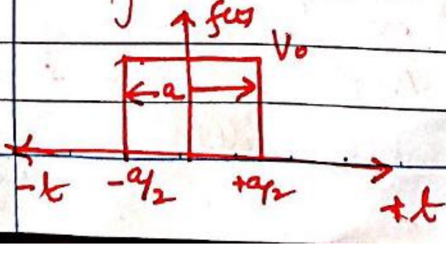
Plot Continuous Frequency & phase spectrum

Q.23. Exclude → OMIT

Q.24 Exclude & OMIT.

Q.25 Using Pulse of Q.22 determine its Bandwidth.

Q.26. For the pulse shown prove that $F(j\omega) = V_0 a \sin(\omega a/2)$ and its BW between first Zeroes is $(2BW) = 2\pi/a$. Also plot the Spectrum.



Solutions / Hints

LECTURE #06 - Topic 2 (Contd).

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TSheet #(02) Contd. → Some Solutions & Some Hints

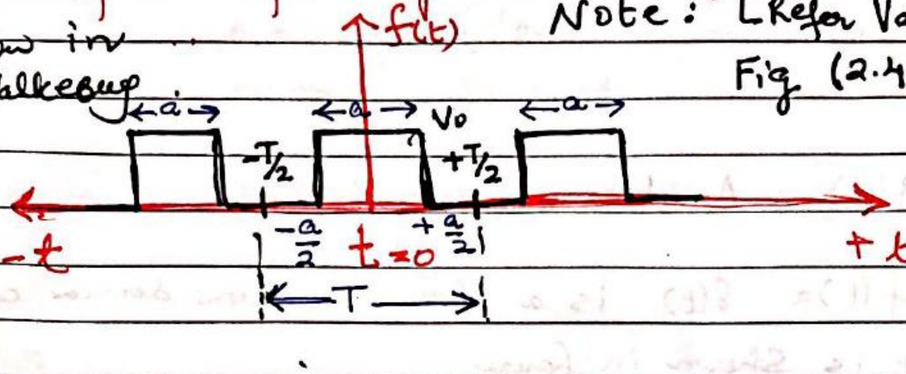
Network Analysis by Van Valkenburg.

Q.17(a) Draw the Double Sided Amplitude Spectrum and the corresponding Double Sided Phase Spectrum of the following Periodic Square waveform

HINT

Solution in Van Valkenburg

Note: [Refer Van Valkenburg]. Fig (2.4):-



Fig(2.4): The figure above shows a square periodic wave form with Amplitude = V_0; Duration = a; Timeperiod T. so that $\omega_0 = 2\pi f$ or $\omega_0 = \frac{2\pi}{T}$; $\frac{a}{T}$ is the duty factor.

Attempt the question for the following cases.

a(i) When $\frac{a}{T} = \frac{1}{2} \Rightarrow T = 2a$

a(ii) $\frac{a}{T} = \frac{1}{4} \Rightarrow T = 4a$

a(iii) $\frac{a}{T} = \frac{1}{5} \Rightarrow T = 5a$

Here for fixed a width Time period is increasing.

Q.17.(b) In general if $\frac{a}{T} = \frac{1}{N}$, then as N increases. what will be influence of change in $\frac{a}{T}$ ration on following [Refer pg. 50 of Lecture #05 dt 29.04.20

HINT

b(i) Spectrum
b(ii) Spacing Δω between components
b(iii) Amplitude of spectrum
b(iv) Zero crossing (First Zero crossing).

~~Q.19~~ Q.19 Find the Fourier Transform of following function and draw the corresponding Frequency and phase spectrum. [Refer Network Analysis by Van Valkenberg].

19(4) A delta function $\delta(t)$
19(5) $f(t) = 1$
19(6) $f(t) = \exp(j\omega_0 t)$
19(7) $f(t) = \exp(-j\omega_0 t)$.

Q. 19(4)

HINT 19(4) A delta function :

$f(t) = \delta(t)$ is a delta in time domain and it is shown in figure

$$\therefore \mathcal{F}(f(t)) = F(j\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j\omega t)\, dt \quad \ldots (2.9.c)$$

$$F(j\omega) = \int_{-\infty}^{\infty} \delta(t) \exp(-j\omega t)\, dt$$

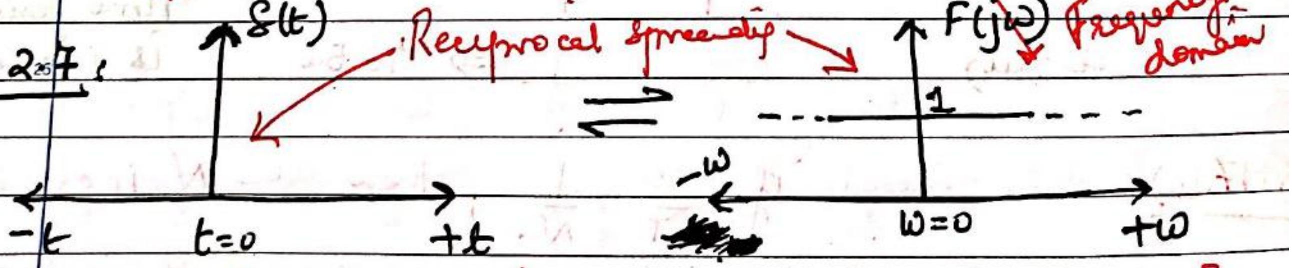
$$= \int_{-\infty}^{\infty} \delta(t-0) \exp(-j\omega t)\, dt$$

$$= \exp[-j\omega(0)] = 1$$

Time domain

$$\therefore \boxed{\delta(t) \rightleftharpoons 1}$$ form a Fourier Transform Pair

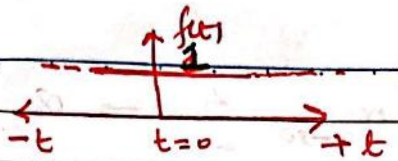
Fig 2.7,



→ An Impulse in time domain → Uniform Spectrum in Frequency domain

Phase Spectrum :- Zero phase contribution ,
The above Transform pair is an example of Reciprocal Spreading. This means an impulse at t=0 requires Infinite number of Components in Frequency domain.

8 19(5) $f(t) = 1$



dc; $\omega_0=0$

Hint f(t)=1 implies that this is a dc of magnitude 1. By inspection and by the knowlege of Frequency spectrum it is clear that dc voltage means Zero frequency. Thus we contempelate that the corresponding frequency domain will have a component at ω=0

$$\mathcal{F}[f(t)] = F(j\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)\,dt$$

or $$F(j\omega) = \int_{-\infty}^{\infty}(1)\exp(-j\omega t)\,dt.$$

We may also write

$$f(t) = 1 = \lim_{a\to 0}\exp[-a|t|]$$

$$\therefore F(j\omega) = \lim_{a\to 0}\int_{-\infty}^{0}\exp(at)\exp(-j\omega t)\,dt + \lim_{a\to 0}\int_{0}^{\infty}\exp(-at)\exp(-j\omega t)\,dt.$$

$$= \lim_{a\to 0}\int_{-\infty}^{0}\exp(a-j\omega)t\,dt + \lim_{a\to 0}\int_{0}^{\infty}\exp^{-}(a+j\omega)t\,dt.$$

On complete solution of the integrals we get

$$\boxed{F(j\omega) = \lim_{a\to 0}\left[\frac{2a}{a^2+\omega^2}\right]}$$

$F(j\omega)\big|_{\omega=0}$ will give the Frequency domain spectrum at ω=0.

However since at ω=0, $F(j\omega)\big|_{\omega=0} = \lim_{a\to 0}\left[\frac{2a}{a^2+\omega^2}\right]$ is of form 0/0 so we apply L Hospital's rule. We Differentiate numerator and denominator wrt a and then apply limit $a\to 0$.

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$$\therefore F(j\omega)\Big|_{\omega=0} = \lim_{a\to 0} \frac{2}{2a} \Rightarrow \infty$$

$F(j\omega) \to \infty$ at $\omega=0$ implies that there exists an impulse $\delta(\omega)$ at $\omega=0$ in frequency domain.

To find magnitude of the impulse ie strength of the impulse we integrate $F(j\omega)$ for all ω to give area contained by Function

$$\text{Strength of Impulse } \delta(\omega) = \int_{-\infty}^{\infty} \frac{2a}{a^2+\omega^2}\, d\omega = 2\pi$$

On solving this gives 2π as the strength of $\delta(\omega)$.

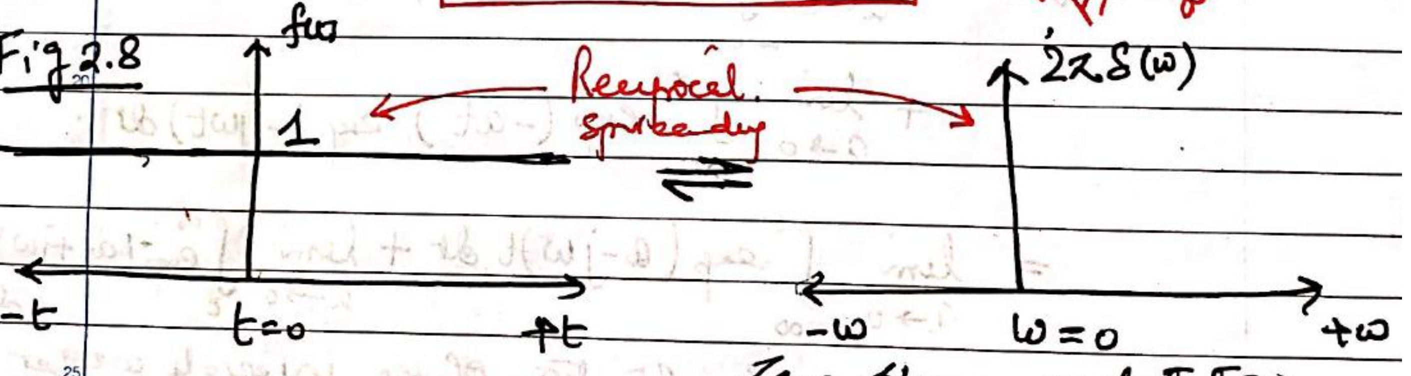
Corresponding $f(t)$ & $F(j\omega)$ are drawn below as indicated from the derived Fourier xform pair

Time domain

$$\therefore \boxed{1 = 2\pi\,\delta(\omega)}$$

Frequency domain

Fig 2.8



dc voltage of magnitude 1 volts

Zero phase contribution

dc voltage in Time domain

Frequency domain shows a non-oscillatory component at $\omega=0$. Impulse of strength 2π in Frequency domain.

This is another example of Reciprocal Spreading because a dc in time domain is shown as a constant line from $t=-\infty$ to $t=+\infty$. whereas its Frequency domain representation will be just an impulse of strength 2π at $\omega=0$. This mean non-oscillatory.

$$\therefore F(j\omega)\Big|_{\omega=0} = \lim_{a\to 0} \frac{2}{2a} \Rightarrow \infty$$

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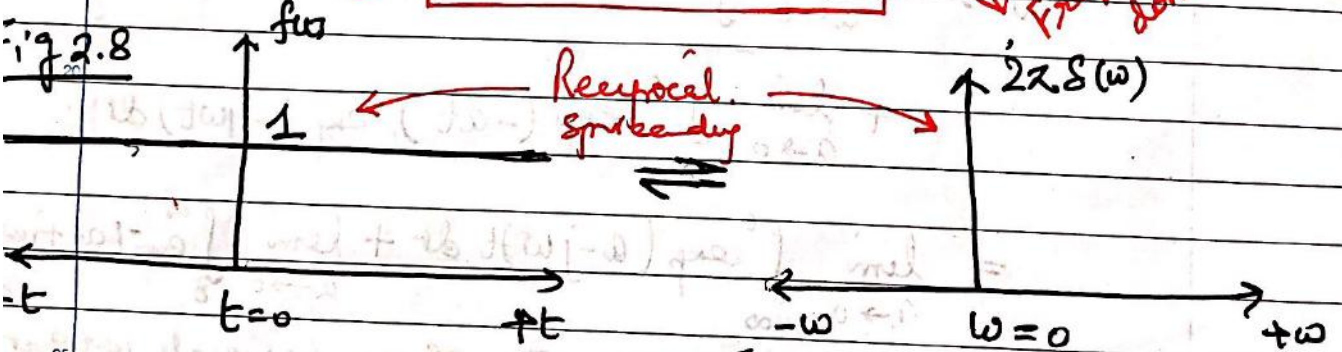
Corresponding $f(t)$ & $F(j\omega)$ are drawn below as indicated from the derived Fourier xform pair

Time domain $\swarrow$ $\boxed{1 = 2\pi\,\delta(\omega)}$ $\rightarrow$ Frequency domain

Fig 2.8



Zero phase contribution

dc voltage of magnitude 1 volts

dc voltage in Time domain

Frequency domain shows a non-oscillatory component at ω=0. Impulse of strength 2π in Frequency domain.

This is another example of Reciprocal Spreading because a dc in time domain is shown as a constant line from $t=-\infty$ to $t=+\infty$ whereas its Frequency domain representation will be just an impulse of strength 2π at $\omega=0$. This mean non-oscillatory.

Thumbs rule # 2.13.

$$\boxed{\delta(t) \rightleftharpoons 1} \quad —(i)$$

$$\boxed{1 \rightleftharpoons 2\pi\delta(\omega)} \quad —(ii)$$

Above two transforms can be summed up as an

(i) impulse in time domain requires infinite components of equal magnitude to build it up.

(ii) A dc in time domain existing from $t = -\infty$ to $+\infty$ requires just an impulse $2\pi\,\delta(\omega)$ to build it up.

The above two Transforms are examples of the property of Reciprocal spreading and Duality. How? Explain. Hint: $f(t) \rightleftharpoons F(\omega)$ then $F(t) \rightleftharpoons 2\pi f(-\omega)$. Here shapes have interchanged.

Q. 19(6) $f(t) = \exp(j\omega_0 t)$. — To find its Transform

$f(t) = \exp(j\omega_0 t)$ can also be written as.

$$f(t) = 1 \cdot \exp(j\omega_0 t).$$

$$\mathcal{F}(j\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j\,\omega t)\,dt \quad --- (2.9.c)$$

$$F(j\omega) = \int_{-\infty}^{\infty} \exp(j\omega_0 t)\exp(-j\omega t)\,dt$$

$$= \int_{-\infty}^{\infty} \exp(-j)(\omega-\omega_0)t\;dt.$$

$$\text{or } F(j\omega) = \int_{-\infty}^{\infty} 1 \cdot \exp[j(\omega-\omega_0)t]\,dt.$$

Since already we have found the following fourier Transform pair

$$\boxed{1 \rightleftharpoons 2\pi\,\delta(\omega)}$$

i.e $\int_{-\infty}^{\infty} 1 . \exp(-j\omega t)dt = 2\pi\,\delta(\omega)$

$\therefore \int_{-\infty}^{\infty} 1 \cdot \exp[-j(\omega - \omega_0)t] = 2\pi\,\delta(\omega-\omega_0).$

This gives us the new Transform pair of

$$\boxed{\exp(j\omega_0 t) \rightleftharpoons 2\pi\,\delta(\omega-\omega_0)}$$

Here we can see that we have just got an application of <u>Frequency shifting property</u>. How? Explain.

Hint:- The frequency shifting Property says

if $f(t) \rightleftharpoons F(\omega)$ then

$f(t)\exp(j\omega_0 t) \rightleftharpoons F(\omega-\omega_0)$,

Here too since $1 \rightleftharpoons 2\pi\,\delta(\omega)$

then $1 . \exp(j\omega_0 t) \rightleftharpoons 2\pi\,\delta(\omega-\omega_0)$.

Q-19(7) $f(t) = \exp(-j\omega_0 t)$, Find its Transform.

Hint: We can directly use above Transform information & result to find the transform of $f(t) = \exp(-j\omega_0 t)$.

So we write..

Since $1 \rightleftharpoons 2\pi\,\delta(\omega)$

$$\exp(j\omega_0 t) \rightleftharpoons 2\pi\,\delta(\omega-\omega_0).$$

Hence $\exp(-j\omega_0 t) \rightleftharpoons 2\pi\,\delta(\omega+\omega_0)$.

Q. 20 Obtain Fourier Trasform with neat illustrations of (1) $f(t) = \mathrm{Cos}(\omega_0 t)$

(2) $f(t) = \mathrm{Sin}(\omega_0 t)$.

Q. 20(1) $f(t) = \mathrm{Cos}(\omega_0 t)$

Hint :-

Here we use Euler's identity / formula as was already given on pg. (32) & pg. (33) of Lecture #04, dt 23-04-20.

$$\mathrm{Cos}(\omega_0 t) = \frac{\exp(j\omega_0 t) + \exp(-j\omega_0 t)}{2}$$

Here Linearity Property is used
How?

$$\therefore \mathcal{F}\left(\mathrm{Cos}(\omega_0 t)\right) = \mathcal{F}\left[\frac{\exp(j\omega_0 t) + \exp(-j\omega_0 t)}{2}\right].$$

or $\mathcal{F}[\mathrm{Cos}(\omega_0 t)] = \frac{1}{2}\mathcal{F}[\exp(j\omega_0 t)] + \frac{1}{2}\mathcal{F}[\exp(-j\omega_0 t)]$

Already we know the transforms

$$\mathcal{F}[\exp(j\omega_0 t)] = 2\pi\,\delta(\omega-\omega_0).$$

&

$$\mathcal{F}[\exp(-j\omega_0 t)] = 2\pi\,\delta(\omega+\omega_0).$$

Using these results

$\therefore \mathcal{F}[\mathrm{Cos}(\omega_0 t)] = \frac{1}{2}[2\pi\,\delta(\omega-\omega_0)] + \frac{1}{2}[2\pi\,\delta(\omega+\omega_0$

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or $$\boxed{\cos(\omega_0 t) \rightleftharpoons \pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0).}$$

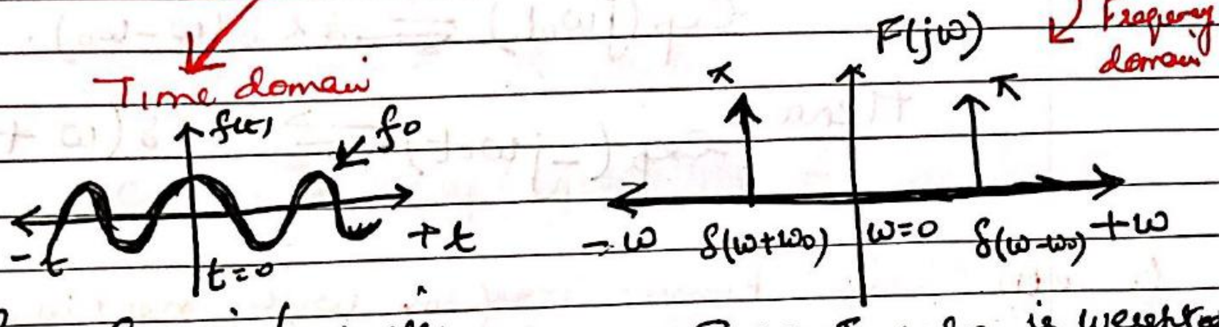


Fig (2.9) Pure Cosine for which $\omega_0 = 2\pi f_0$ angular freq.

Each Impulse is weighted by π or is of strength π. Since F(jω) obtained is real ∴ Zero phase Spectrum

Q. 20(2) $f(t) = \sin(\omega_0 t)$.

Hint : Again using Euler's Formula as mentioned on page (32) & (33) whereby.

$$\sin(\omega_0 t) = \frac{\exp[j\omega_0 t] - \exp[-j\omega_0 t]}{2j}$$

Here Linearity Property used. How?

$$\therefore \mathcal{F}[\sin(\omega_0 t)] = \frac{1}{2j}\mathcal{F}[\exp(j\omega_0 t)] - \frac{1}{2j}\mathcal{F}[\exp(-j\omega_0 t)]$$

Using results already found wherein

$$\mathcal{F}[\exp(j\omega_0 t)] \rightleftharpoons 2\pi\delta(\omega-\omega_0)$$

$$\mathcal{F}[\exp(-j\omega_0 t) \rightleftharpoons 2\pi\delta(\omega+\omega_0)$$

So we may use these Transforms

$$\mathcal{F}[\sin(\omega_0 t)] \rightleftharpoons \frac{1}{2j}[2\pi\delta(\omega-\omega_0)] - \frac{1}{2j}[2\pi\delta(\omega+$$

or $$\boxed{\sin(\omega_0 t) \rightleftharpoons \frac{\pi}{j}\delta(\omega-\omega_0) - \frac{\pi}{j}\delta(\omega+\omega_0)}$$

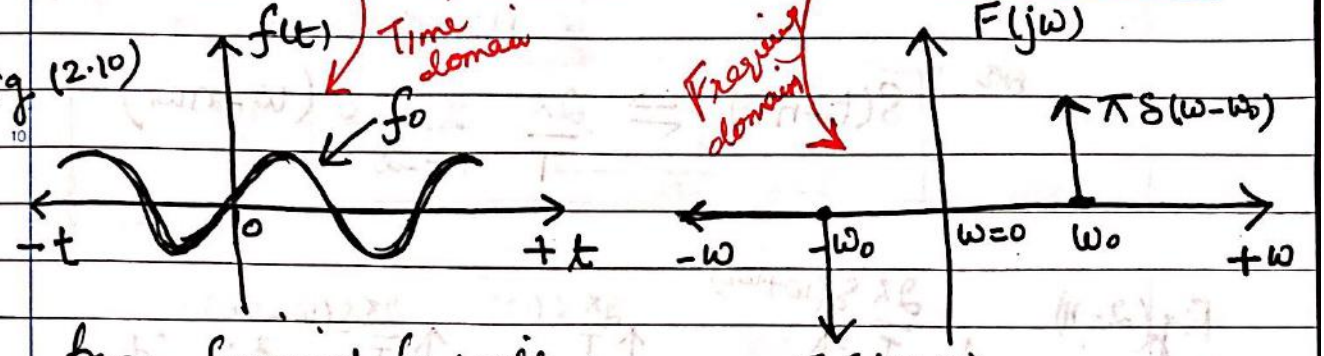
Here Since $F(j\omega)$ is imaginary ∴ Phase spectrum will exist. Now we have to draw the illustrations in Time & Frequency domain.

$$\sin(\omega_0 t) \rightleftharpoons \frac{\pi}{j}\,\delta(\omega-\omega_0) - \frac{\pi}{j}\,\delta(\omega+\omega_0)$$

or

$$\boxed{\sin(\omega_0 t) \rightleftharpoons -j\pi\,\delta(\omega-\omega_0) + j\pi\,\delta(\omega+\omega_0).}$$

Fig (2.10)



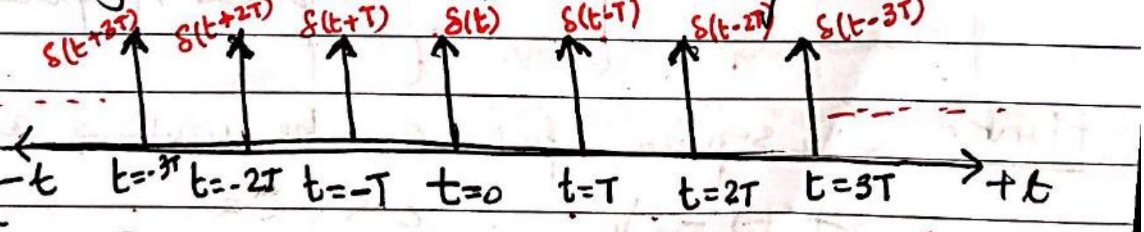
pure sinusoid fn with
$\omega_0 = 2\pi f_0$ angular frequency

Above Transform show that here Exists a phase contribution which is understood because there is a $\pi/2$ phase difference with a pure cosine reference. Also it is clear hence phas associated with two fns $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ differ by $\pi/2$

Q-21 Determine the Fourier Transform of an Impulse train

Hint $f(t) = \delta(t-nT)$ is an impulse train

Impulse Train in Time domain



Since $C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\exp[-jn\omega_0 t]\,dt$

or $C_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\exp[-jn\omega_0 t]\,dt$ ·

Using previously derived Fourier Transform where

$$\exp(j\omega_0 t) \rightleftharpoons 2\pi\,\delta(\omega-\omega_0)$$

&

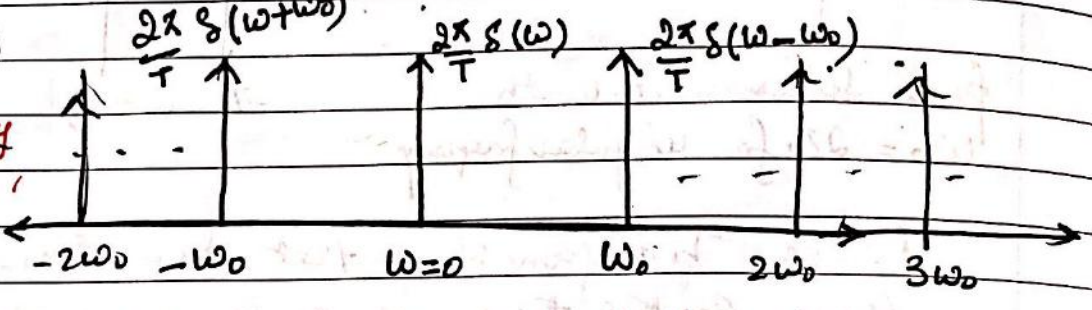
$$\exp(-j\omega_0 t) \rightleftharpoons 2\pi\,\delta(\omega+\omega_0)$$

$$\therefore \mathcal{F}[\delta(t-nT)] = 2\pi \sum_{n=-\infty}^{\infty} C_n\,\delta(\omega-n\omega_0)$$

or

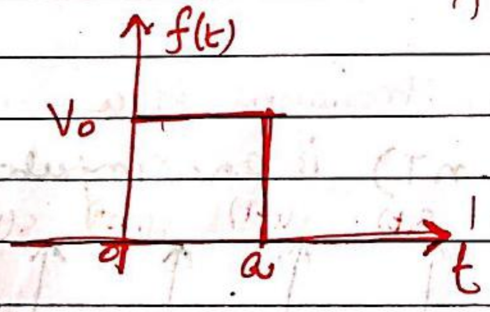
$$\boxed{\delta(t-nT) \rightleftharpoons \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega-n\omega_0)}$$

Fig (2.11)
Frequency domain



Thus Fourier Transform of an impulse Train in Time domain is an impulse Train in Frequency Domain

Q. 22 Obtain Fourier Transform and plot Frequency Spectrum



pulse Width = a
pulse Amplitude = V_0

Hint:- The square pulse can be written as

$f(t) = V_0 \ldots \quad 0 < t < a$

$= 0$ for $t < 0$ and $t > a$

Now we apply $F(j\omega) = \mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)\,dt$.

$$\therefore F(j\omega) = \int_0^a V_0 \exp(-j\omega t)\,dt$$

After solving completely the solution is

Ans →
$$\boxed{F(j\omega) = \frac{V_0}{\omega}\sin(\omega a) - jV_0/_{\omega}\,(1-\cos\omega a).}$$

Now as F(jω) is complex so we may also write

$$F(j\omega) = \underset{\text{Magnitude}}{|F(j\omega)|}\exp[j\underset{\text{Phase}}{\varphi(\omega)}] \text{ --- } (2.9.e)$$

After using formula we obtain magnitude

$$\boxed{|F(j\omega)| = V_0 a\left(\frac{\sin(\omega a/_2)}{\omega a/_2}\right)} \rightarrow \text{Sinc fn}$$

$\therefore |F(j\omega)| = 0$ for $\omega = \pm\frac{2n\pi}{a}$

$\Rightarrow \omega = \pm 2\pi/a;\ \pm 4\pi/a;\ \pm\frac{6\pi}{a}\ldots$ are positions of zeroes.

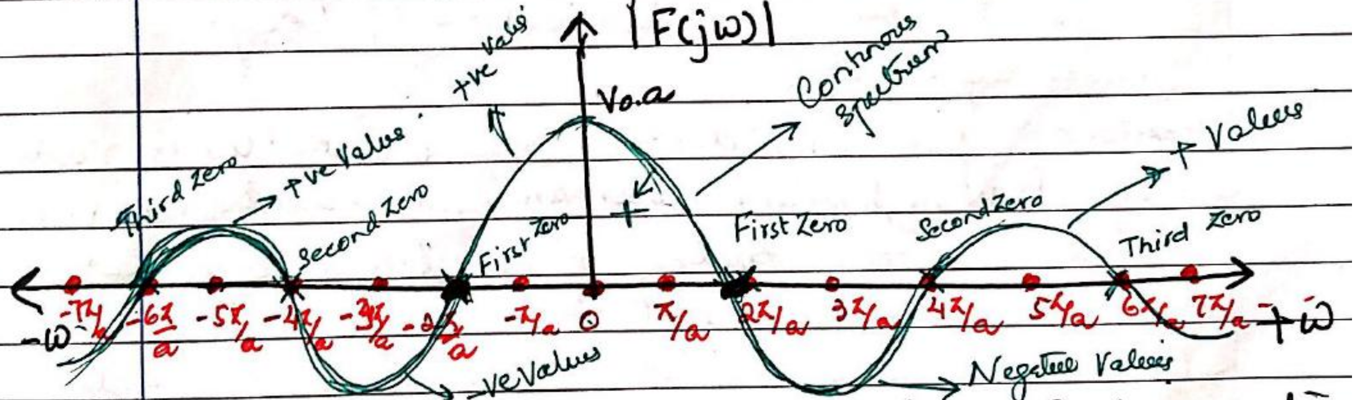
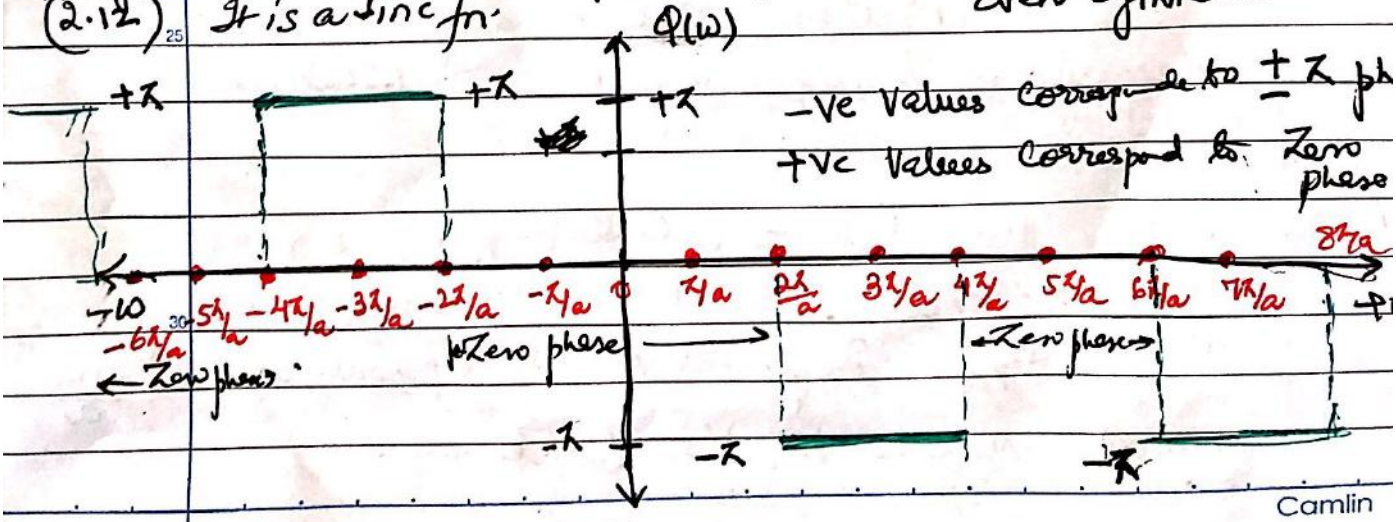


Fig (2.12) Double sided Amplitude spectrum: It is a Continuous spectrum Even Symmetric.
It is a sinc fn.



-ve Values correspond to ± π ph
+ve Values correspond to Zero phase

Bandwidth Calculation :- (BW)

The transmission bandwidth is the Bandwidth Covered by the function f(t). As the Frequency spectrum shows extension from $\omega \rightarrow +\infty$ to $\omega \rightarrow \infty$ thus B.W covered is ∞.

However we can Truncate the B.W covered as the limits ~~for~~ from $\omega = -2\pi/a$ to $\omega = +2\pi/a$. Thus approximately Signal Bandwidth is calculated b/w First Zero crossings.

$\therefore$ Signal(2BW) = $4\pi/a$ $(-2\pi/a$ to $+2\pi/a)$

Since actually only +ve values of frequency matter and exist therefore

$$\boxed{B.W = 2\pi/a}$$

This is yet another implication of Reciprocal Spreading. How?

Because if in time domain pulse width (a) is reduced then B.W in frequency domain increases & vice Versa. So Narrower the pulse, more B.W is required to efficiently transmit the pulse. Thus in Communication channel Large pulse widths are preferred so that BW required is small. But there are many other demerits of using large B.W - ~~what are~~ What are those?

Q.26 HINT here $F(j\omega) = V_0\, a\, \text{Sin}(\omega a/2)$ is real so no phase spectrum. First Zero crossing will be $\omega = \pm 2\pi/a$. In general Zero crossing will at $\omega = \pm \frac{2n\pi}{a}$

Complete Solution In Van Valkenburg