L-10

# A.C. Circuit Analysis or Sinusoidal Steady State Analysis



The sine wave is one of the most widely encountered and used waveforms in electrical engineering. Easily generated by ac machines it is distributed by transmission lines and brought to our homes to run a variety of analysis. In addition it is generated by laboratory oscillators to test the performance of circuits. Mathematically, the sine wave is described by a simple function having the form A cos($\omega t + \theta$) or A sin($\omega t + \theta$). This function on differentiation or integration yields a sinusoidal function. Furthermore, the sum of two or more sine waves of the same frequency is also a sine wave. Consequently when Kirchoff's laws are applied to sum the sinusoidal currents or voltages, the unknown variables can be easily determined.

Because the sine wave is important and simple to deal with, a great deal of knowledge exists about sinusoidally excited networks.

The theory of the sinusoidal steady-state response circuits occupies a position of pre-eminence in electric-circuit theory. The analysis of many circuits and devices throughout all branches of electrical engineering is accomplished by the techniques in the sinusoidal theory. Thus, the study of circuits with sinusoidal sources is a central theme in electrical engineering.

## Sinusoidal functions - terminology

A sinusoidal voltage source produces a voltage that varies sinusoidally with time. A sinusoidal current source produces a current that varies sinusoidally with time.

In reviewing the sinusoidal function let us use a voltage. We can express a sinusoidal voltage either the sine function or cosine function. Although either works equally well, ~~but~~ we cannot use both function forms simultaneously. We will use the cosine functions.

A sinusoidal voltage as a cosine function has the form

$$v(t) = V_m \cos(\omega t + \theta)$$

where $V_m$ is the peak value of the voltage, $\omega$ is the angular frequency in radians/sec and $\theta$ is the phase angle.

Sinusoidal functions are periodic, repeating the same pattern of values in each period T. It is measured in seconds. The reciprocal of T gives the number of cycles per second, or the frequency 'f' of the sinewave. Thus we have

$$f = \frac{1}{T}$$

The units of frequency are hertz (Hz). Since the cosine (or sine) function completes one cycle when the angle increases by $2\pi$ radians, we get

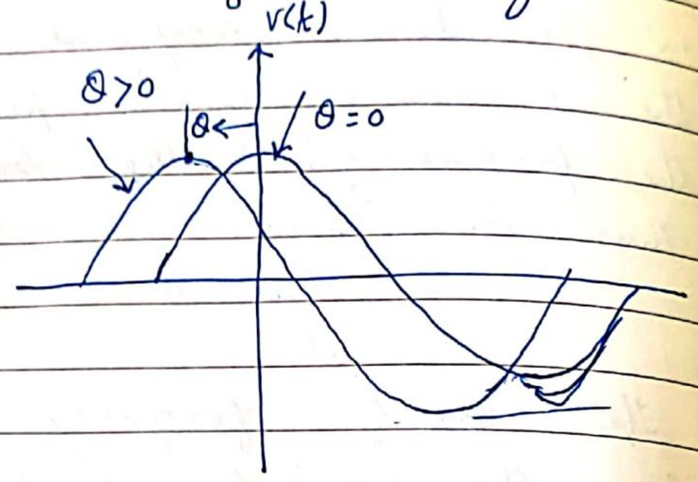
$$\omega T = 2\pi$$

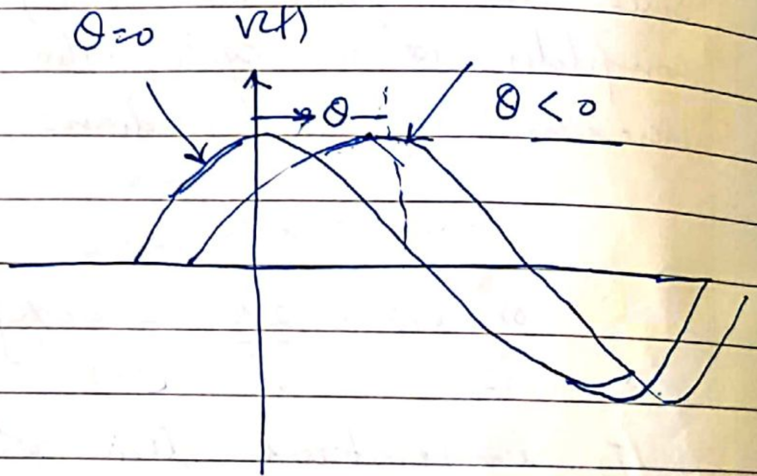
$$\text{or } \omega = \frac{2\pi}{T} = 2\pi f$$

To understand the role of the phase angle, recall that the cosine function reaches its maximum value when its argument is an integral multiple of $2\pi$, that is when $\omega t + \theta = \pm 2\pi n$, when n is an arbitrary integer. The maximum corresponding to $n = 0$, referred to as

central peak occurs at $\omega t + \theta = 0$
that is when $\omega t = -\theta$

Clearly with $\theta = 0$, the central peak occurs at $t = 0$. However when $\theta > 0$, the central peak waveform shifts towards left. Conversely if $\theta < 0$, the central peak waveform shifts towards right.




In the argument of cosine (or sine function) $\omega t + \theta$ we assume that angular frequency $\omega$ has units of rad/s. However we usually give the phase angle $\theta$ is degrees as we find it easier to visualize an angle expressed in degrees and mixed units are not a problem. If we want to evaluate $\cos(\omega t + \theta)$ for any particular value of time, we would have to convert $\theta$ to radians before adding the terms in the argument.

Another import characteristics of the sinusoidal voltage (or current) is its rms value. The rms value of a periodic function is defined as the square root of the mean value of the squared function. Hence if $v(t) = V_m \cos(\omega t + \theta)$, the rms value of $v$ is

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \cos^2(\omega t + \theta)\, dt}$$

$$= \frac{V_m}{\sqrt{2}}.$$

The rms value of the sinusoidal voltage depends only on the maximum value of $v(t)$, namely $V_m$.

Root-mean square (RMS) values are often used in electrical engineering to measure voltages and currents. Apart from providing the amplitude information use of rms values offer some computational advantages, especially when dealing with power. To see this, we compute the average power dissipation in a resistance R.

Assume that the voltage across this resistor is $v(t)$ and the current is $i(t)$. From the definition of average power, we have

$$P_{av} = \frac{1}{T}\int_0^T \frac{v^2(t)}{R}\,dt$$

$$= \frac{V_{rms}^2}{R}$$

$$\text{or } P_{av} = \frac{1}{T}\int_0^T i^2(t)\,R\,dt$$

$$= I_{rms}^2 R.$$

Clearly, the above formulae resemble those of the DC case. A moment's thought will convince us that as far as power is concerned, a circuit driven by AC sources can be analyzed as if it is driven by DC sources whose values equal those of rms values of the corresponding AC sources.

# Phasor representation of Sinusoids

Before taking up the phasor representation of Sinusoids, let us try finding $y = y_1 + y_2$ (using trigonometric identities) where

$$y_1 = 20\cos(\omega t - 30^\circ)$$

and $$y_2 = 40\cos(\omega t + 60^\circ)$$

$$y_1 = 20\cos(\omega t - 30^\circ)$$
$$= 20\cos\omega t\cos 30 + 20\sin\omega t\sin 30$$

$$y_2 = 40\cos(\omega t + 60)$$
$$= 40\cos\omega t\cos 60 - 40\sin\omega t\sin 60$$

~~y = 20cosωtcos (20cosωt~~

$$y = (20\cos 30 + 40\cos 60)\cos\omega t$$
$$+ (20\sin 30 - 40\sin 60)\sin\omega t$$

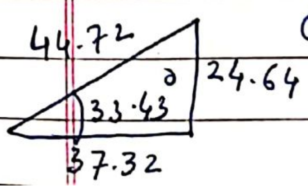
$$y = 37.32\cos\omega t - 24.64\sin\omega t$$

Let $37.32 = A\cos\theta$
and $24.64 = A\sin\theta$

$$y = A\cos(\omega t + \theta)$$

Where $A = \sqrt{(37.32)^2 + (24.67)^2} = 44.72$

$$\theta = \tan^{-1}\left(\frac{24.64}{37.32}\right) = 33.43^\circ$$



$\therefore y = 44.72 \cos(\omega t + 33.43^\circ)$

$\bar{Y}_1 = 20 \angle -30^\circ$

$\bar{Y}_2 = 40 \angle 60^\circ$

$\} \quad \bar{Y} = 44.72 \angle 33.43^\circ$

$\bar{Y}_1 + \bar{Y}_2 = 20 \angle -30^\circ + 40 \angle 60^\circ$

$= (17.32 - j10) + (20 + j34.64)$

$= 37.32 + j24.64$

$= 44.72 \angle 33.43^\circ$

$= \bar{Y}$

## Phasor

The phasor is a complex number that carries the amplitude and phase angle information of a sinusoidal function

For a sinusoidal voltage

$$v(t) = V_m \cos(\omega t + \theta)$$

we define phasor as

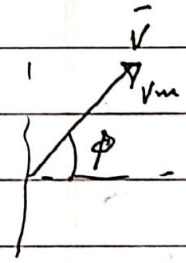
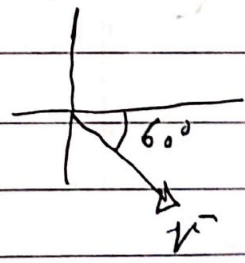
$$\bar{V} = \mathcal{R}\,\mathcal{P}\left(V_m \cos(\omega t + \theta)\right) = V_m e^{j\theta}$$

$$= V_m \angle\theta$$

Actually engineers are not consistent in choosing the magnitude of the phasor.

It is taken as $V_m$ or $V_{rms}$. In case we will take rms value, we will label it otherwise we assume that they are peak values.

Thus we have

| Time domain representation | Phasor representation | |
|---|---|---|
| $V_m \cos(\omega t + \phi)$ | $\bar{V} = V_m \angle\phi$ |  |
| $V_m \sin(\omega t + \phi)$ ↓ 30 | $\bar{V} = V_m \angle\phi - 90^\circ$ |  |

# Inverse phasor transform

We may also reverse the process. That is for a phasor we may write expression for sinusoidal function.

Finding the inverse phasor transform is formalized by the expression

$$\varphi^{-1}\left(V_m e^{j\phi}\right) = \mathcal{R}\left(V_m e^{j\phi} e^{j\omega t}\right)$$

$$= V_m \cos(\omega t + \phi)$$

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$v_1(t) = 20\cos(\omega t - 45^0)$

$v_2(t) = 10\sin(\omega t + 60^0)$

Find $v(t) = v_1(t) + v_2(t)$

$\bar{V_1} = 20\angle{-45^0}$

$\bar{V_2} = 10\angle{60-90} = 10\angle{-30^0}$

$\bar{V} = \bar{V_1} + \bar{V_2}$

$= 14.14 - j14.14 + 8.660 - j5$

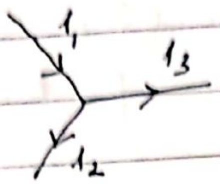
~~$= 22.80 - j19.14$~~

$= 29.77\angle{-40.07^0}$

$\therefore v(t) = 29.77 \cos(\omega t - 40.01^\circ)$

## Kirchoff's Laws in Phasor form:

**KCL :**




$$i_1 = i_2 + i_3 \quad \text{—— (1)}$$

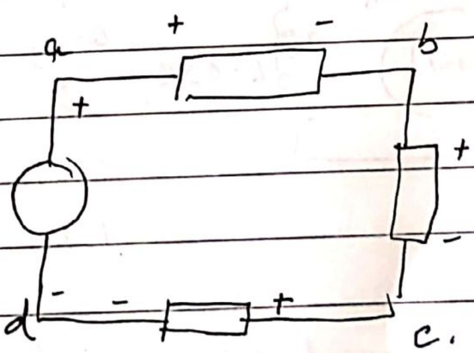
If the currents are sinusoidal they can be represented by phasors.

Then (1) becomes

$$\bar{I}_1 = \bar{I}_2 + \bar{I}_3$$

we can apply KCL directly to phasors

**KVL**

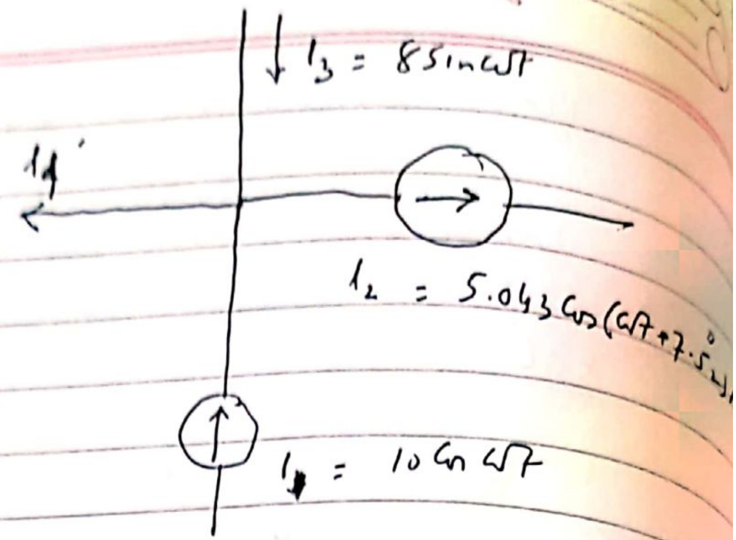



$$V_{ad} = V_{ab} + V_{bc} + V_{ca}$$

For sinusoidal voltages we can write

$$\bar{V}_{ad} = \bar{V}_{ab} + \bar{V}_{bc} + \bar{V}_{ca}$$

we can apply directly KVL to phasors




Find $i_4$

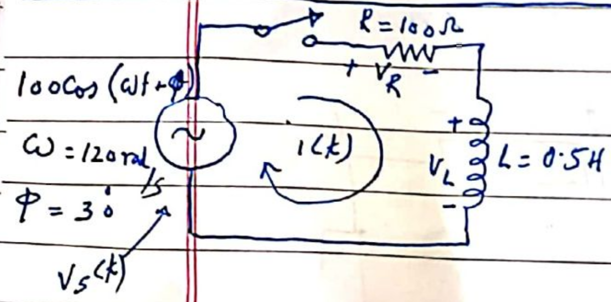
$$i_4 = i_1(t) - i_2(t) + i_3(t)$$

$$= 10\cos(\omega t) - 5.043\cos(\omega t + 7.52) + 8\cos(\omega t - 90)$$

$$\bar{I}_4 = 10 - 5.043\angle 7.52 + 8\angle -90^\circ$$

$$= 10\angle -60$$

$$\therefore i_4(t) = 10\cos(\omega t - 60^\circ)$$




After switch is closed we apply KVL: $V_s(t) = V_R(t) + V_L(t)$

or $L\dfrac{di}{dt} + Ri = V_p\cos(\omega t + \phi)$

$i(t) = Ae^{-t/\tau} + i_{ss}(t)$

$i(t) = -0.397e^{-200t} + 0.469\cos(120\pi t - 32.1^\circ)$

The current $i(t)$ rapidly approches the sinusoidal steady-state part of the solution after the switch is closed.

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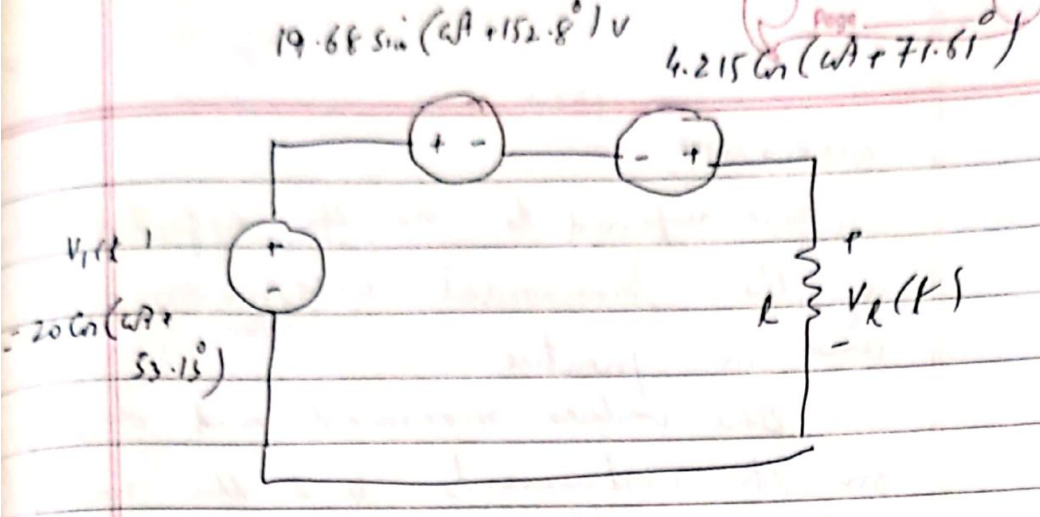


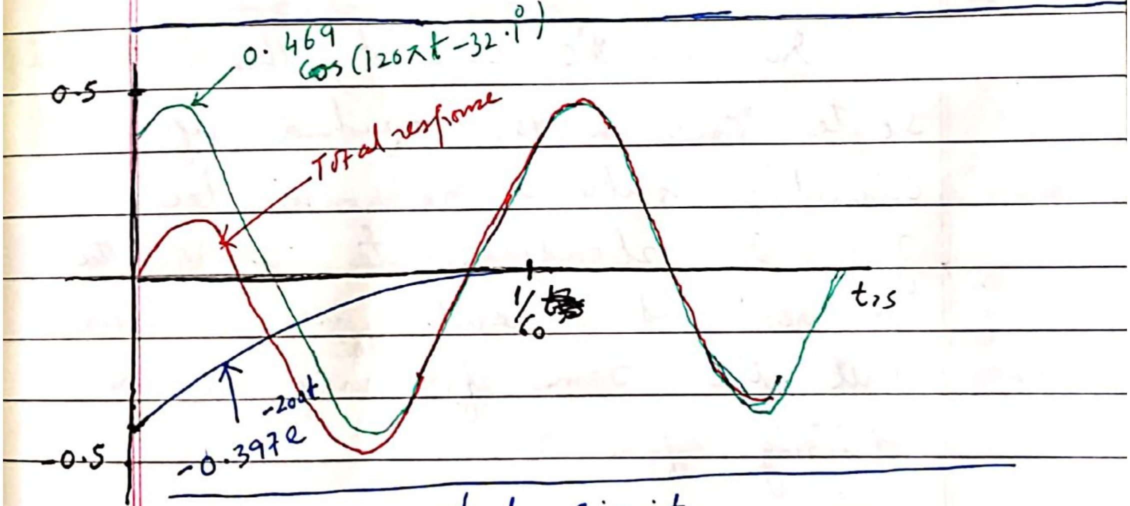

Find $V_R(t)$

$$V_R(t) = V_1(t) - V_2(t) + V_3(t)$$

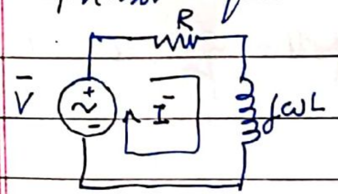
$$\bar{V}_R = 20\angle 53.13^\circ - 19.68\angle 62.8^\circ + 4.215\angle 71.6^\circ$$

$$\bar{V}_R = 5\angle 30^\circ$$

$$\therefore V_R(t) = 5\cos(\omega t + 30^\circ)\ V$$




Phasor equivalent circuit:




$$\bar{I} = \frac{\bar{V}}{R + j\omega L} = \frac{100\angle 30^\circ}{100 + j(120\pi \times 0.5)}$$

$$= \frac{100\angle 30^\circ}{213\angle 62.1^\circ} = 0.469\angle -32.1^\circ$$

$$\therefore I(t) = .469\cos(120\pi t - 32.1^\circ)$$

RMS value

(also referred to as the effective value) of the sinusoidal waveform is widely used in practice.

The values measured and displayed on the instruments and the nominal ratings of the equipment are rms values.

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## Connection Laws in Phasor Form

We can find the phasor for a sum of sinusoids having same frequency but different amplitudes and phase angles by summing the individual phasors using complex arithmetic.

In an AC circuit, the sinusoidal state condition is reached after the circuit's natural response decays to zero. In steady state all of the voltages and currents are sinusoids with the same frequency as the driving force.

Consider the following circuit

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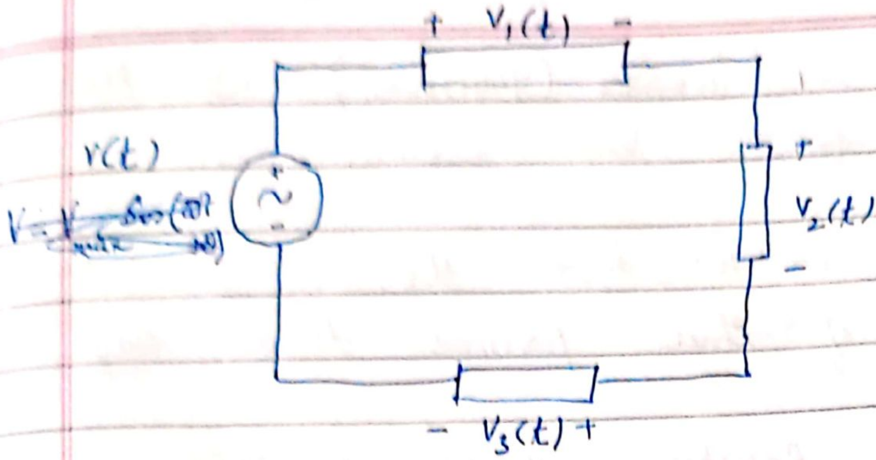


As per KVL

$$V(t) = V_1(t) + V_2(t) + V_3(t)$$

If $v(t) = V_{max} \cos(\omega t + \theta)$

then in sinusoidal steady state we have

$$V_{max} \cos(\omega t + \theta) = V_{1max} \cos(\omega t + \phi_1) + V_{2max} \cos(\omega t + \phi_2) + V_{3max} \cos(\omega t + \phi_3)$$

The corresponding phasor form will be

$$\bar{V} = \bar{V}_1 + \bar{V}_2 + \bar{V}_3$$

This indicates KVL holds for phasor voltages just the way it holds in the time domain.

Clearly the same result applies to phasor currents and KCL.

In otherwords, we can state Kirchoff's laws in Phasor form as follows:

KVL: The algebraic sum of phasor voltages around a loop is zero

KCL: The algebraic sum of phasor currents at a node is zero.

Phasor circuit analysis of RLC circuits

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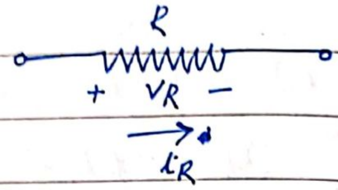
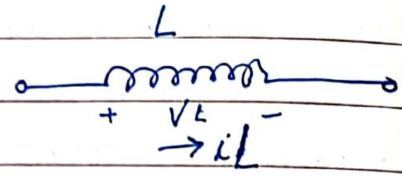
Elements (passive) in Phasor form.

we have the i-v characteristics of three passive devices as:

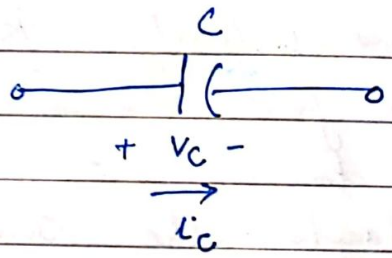
Resistor : $V_R(t) = R\,i_R(t)$




Inductor : $V_L(t) = L \frac{di_L(t)}{dt}$




Capacitor : $i_C(t) = C \frac{dv_C(t)}{dt}$



In the Sinusoidal steady state, all of these current and voltages are sinusoids.

Let us see $\bar{V}-\bar{I}$ relationship for each passive element.

## Resistor

If the current through resistor in steady state is

$$i_R = I_m \cos(\omega t + \theta)$$

then voltage across it is given by

$$V_R = R I_m \cos(\omega t + \theta).$$

The phasor form of current and voltage is

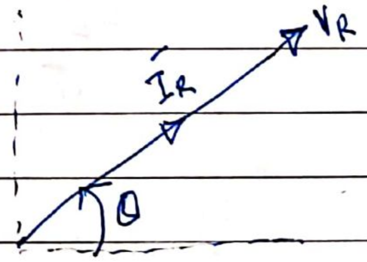
$$\bar{I_R} = I_m \angle \theta$$

and $\bar{V_R} = R I_m \angle \theta$

The phasor diagram is depicted in following figure which illustrates that voltage and current are in phase

$\bar{V}-\bar{I}$ relationship

$\bar{V_R} = R\bar{I_R}$



## Inductor

For the inductor L, assume the current through it is

$$i_L = I_{max}\cos(\omega t + \theta)$$

The voltage across it is

$$v_L = L\frac{di_L}{dt}$$

$$= -\omega L I_{max}\sin(\omega t + \theta)$$

Since $-\sin(A) = \cos(A + 90^\circ)$, we can write

$$v_L = \omega L I_{max}\cos(\omega t + \theta + 90^\circ)$$

In phasor form

$$\bar{I_L} = I_{max}\angle\theta$$

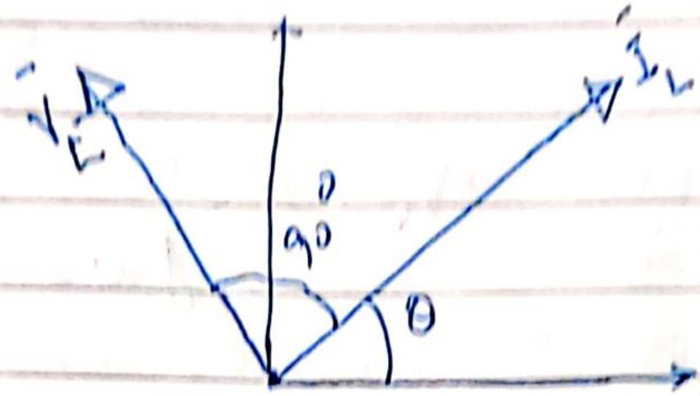
$$\bar{V_L} = \omega L I_{max}\angle\theta + 90^\circ$$

$$\frac{\bar{V_L}}{\bar{I_L}} = \frac{\omega L I_{max}}{I_{max}}\angle 90^\circ$$

$$= \omega L e^{j\pi/2}$$

$$\frac{\bar{V_L}}{\bar{I_L}} = j\omega L$$

$$\bar{V_L} = j\omega L\bar{I}$$




## Capacitor

For the capacitor, assume that voltage across it is

~~$\bar{V_L} = L \frac{d i_L(t)}{dt}$~~

$$V_C(t) = V_{max} \cos(\omega t + \theta)$$

Then current through it is

$$i_C(t) = -\omega C V_{max} \sin(\omega t + \theta)$$

$$= \omega C V_{max} \cos(\omega t + \theta + 90^0)$$

in phasor form

$$\bar{V_C} = V_{max} \angle \theta$$

$$\bar{I_C} = \omega C V_{max} \angle \theta + 90^0$$

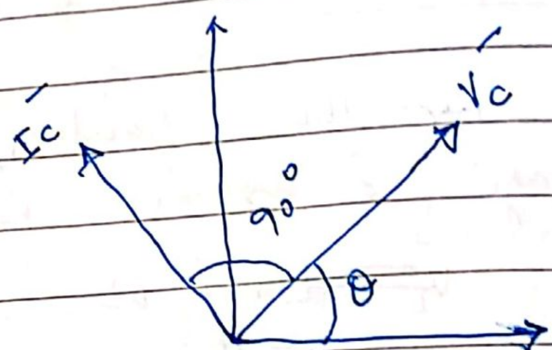
$$\frac{\bar{V_C}}{\bar{I_C}} = \frac{V_{max}}{\omega C V_{max}} \frac{\angle \theta}{\angle \theta + 90^\circ}$$

$$= \frac{1}{\omega C} e^{-j\pi/2}$$

$$= \frac{-j}{\omega C} = \frac{1}{j\omega C}$$

$$\bar{V_C} = \frac{1}{j\omega C} \bar{I_C}$$

Phasor diagram :




Current leads voltage by $90^\circ$

The phasor $\bar{I}-\bar{V}$ constraints for all the three passive elements has the form

$$\bar{V} = Z \bar{I}$$

Where Z is called the impedance of the element.

For resistor $Z_R = R$

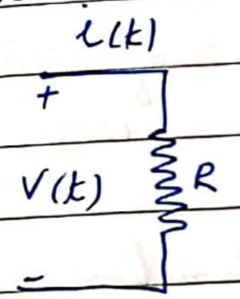
,, Inductor $Z_L = j\omega L$

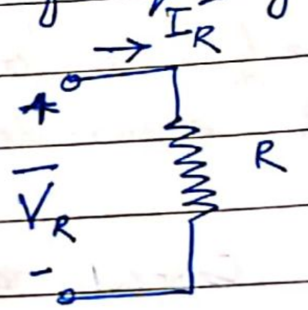
,, Capacitor $Z_C = \frac{1}{j\omega C} = \frac{-j}{\omega C}$

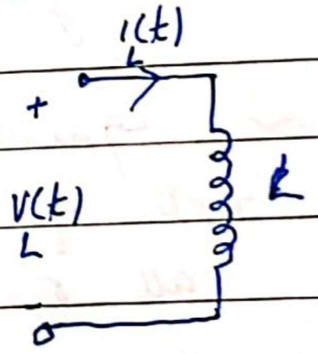
Since impedances relates phasor voltage to phasor current, it is a complex quantity whose units are ohms. Although impedance can be a complex number, it is not a phasor. Phasors represent sinusoidal signals, while the impedances characterise circuit elements in the sinusoidal steady state.

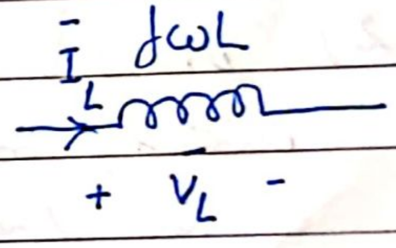
## voltage current relations

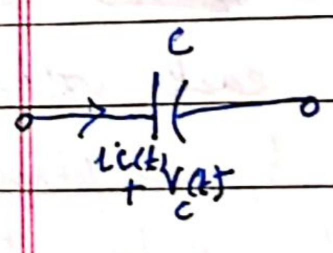
Time domain — frequency domain

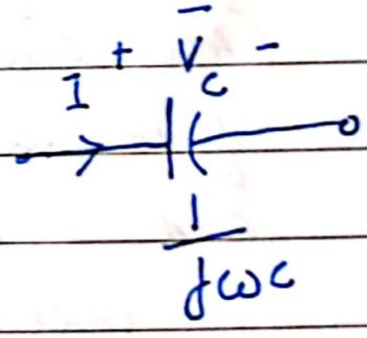


Here is the net result of what we have accomplished. We now know that each of the elements R, L and C obeys Ohms law, provided we use impedance of the appropriate element in the place of resistance. Furthermore we now know that both KVL and KCL hold for phasor voltages and currents. Moreover R, L and C are all linear elements. Thus all of the dc analysis techniques continue to hold for ac steady state analysis* with impedances replacing resistances and phasors replacing time-varying voltages and currents.

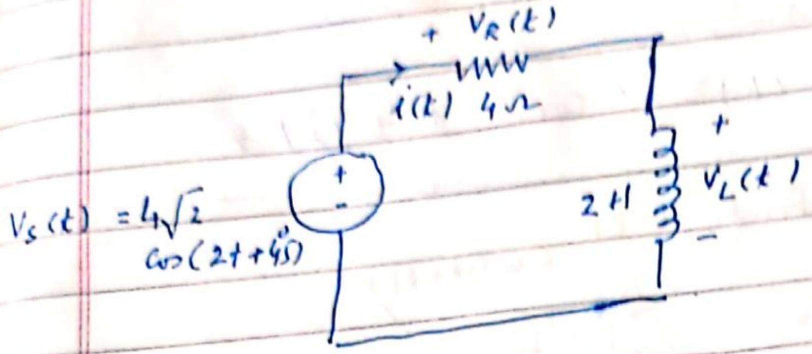
Step-by-step procedure for steady-state analysis of circuits with sinusoidal sources is:

1) Convert all sine functions to cosine functions (if necessary)

2) Draw the phasor equivalent circuit, making a note of the common frequency of all sources (All sources must have same frequency). Represent each voltage and each current by a phasor and each passive element by its impedance

3. Solve for desired phasor(s)

4. Write the time domain for the desired variables.

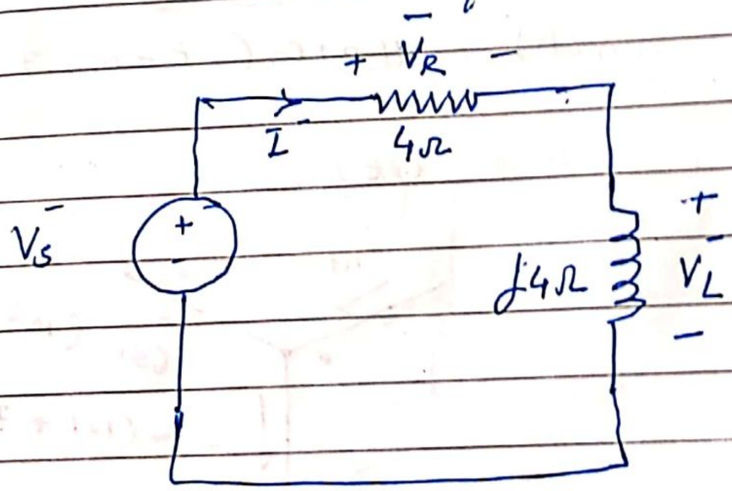



Find $V_L(t)$

$\omega = 2$

$j\omega L = j \times 2 \times 2 = j4\ \Omega$

Phasor equivalent circuit




$\bar{V}_L = \dfrac{j4}{4 + j4} \times \bar{V}_S$ , $\bar{V}_L = \dfrac{j}{1 + j} \bar{V}_S$

$V_s = 4\sqrt{2}$

$$V_L = \frac{1\angle 90 \times 4\sqrt{2}\angle 45^\circ}{\sqrt{2}\angle 45}$$

$$= 4\angle 90^\circ$$

$\therefore V_L(t) \quad 4\cos(2t + 90^\circ)\ V$