## Z- Transform :-

| Continous | Discrete |
|---|---|
| $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$ <br> CTFT | $X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$ <br> DTFT |
| $X(s) = \int_{t=-\infty}^{\infty} x(t) e^{-st}\, dt$ | $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$ <br> z-transform |

DTFT or CTFT → freq. domain rep.

Lap. → gen. of F.T. Replace($j\omega$) by s

Discrete Time version of Laplace x form

Earlier (in DTFT) my $\omega$ was only a real valued generic freq. variable
now, $z$ is some complex no.

<u>Recall:-</u>

$$e^{j\omega_0 n} \longrightarrow \boxed{H} \longrightarrow H(\omega_0)\, e^{j\omega_0 n}$$

In a ~lar way,

$z^n$ is "special" for DT LTI systems

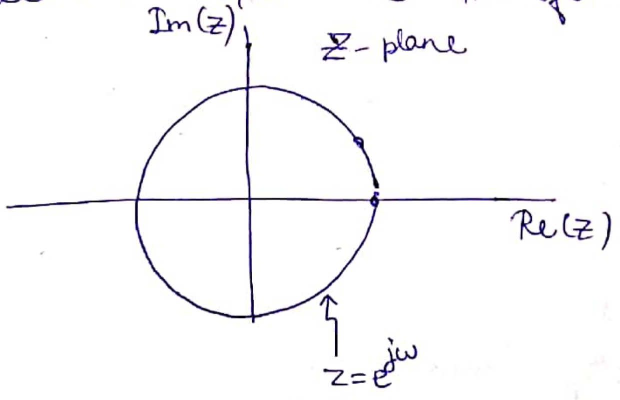
Let $x[n] = z^n$ for some complex $z$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}$$

$$= z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}} = \underbrace{H(z)}_{\text{Transfer function}} z^n$$

& what we show is that if this complex exp. comes into the system what comes out is the very same complex exp. just mul. by this transfer funcn. evaluated at that complex no. $z$.

Relation b/w Z-transform and DTFT:-




$$X(z) = \sum_{n=-\infty} x[n]\, z^{-n}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= X(z)\Big|_{z=e^{j\omega}}$$

DTFT (see the circle) is $2\pi$ periodic)

Formula $X(z)$ is defined for all $z$ ROC.

- unit circle is key for discrete-time systems (like $j\omega$-axis was for CT systems $\} \rightarrow$ when do poles eg. cross over the $j\omega$ axis $\rightarrow$ stability)
  ~ly here, when do poles stay inside the unit circle

Why do we need Z-transform?

- The DTFT doesn't always converge/exist
  (Sufficient : $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ )

- Z-Transform may converge in places where the DTFT doesn't exist
- notation is easier - polynomials in z / rational functions of z.
- helps a lot when designing filters

→ Region of Convergence (ROC)

Let $z = re^{j\omega}$ (Polar form)

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(x[n]\, r^{-n}\right) e^{-j\omega n}$$

$$= \text{DTFT}\left(x[n]\, r^{-n}\right)$$

DTFT of original signal modified by $r^{-n}$

⟺ Z-Transform converges if

$$\sum_{n=-\infty}^{\infty} |x[n]\, r^{-n}| < \infty$$

What I'm doing is to see what is the range of r for which this sum converges.

eg. $u[n]$ : $\sum_{n=-\infty}^{\infty} u[n] = \infty$

So, I cannot take FT of step function

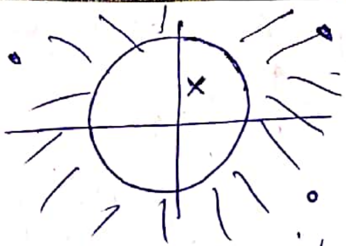
However, what $u[n]\, r^{-n}$ for some real r

$$\sum_{n=-\infty}^{\infty} u[n] \cdot r^{-n} = \sum_{n=0}^{\infty} r^{-n} = \frac{1}{1-r^{-1}} \quad \boxed{\text{if } |r|>1}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{r}\right)^n$$

(give to make sure that each of the terms is less than 1)

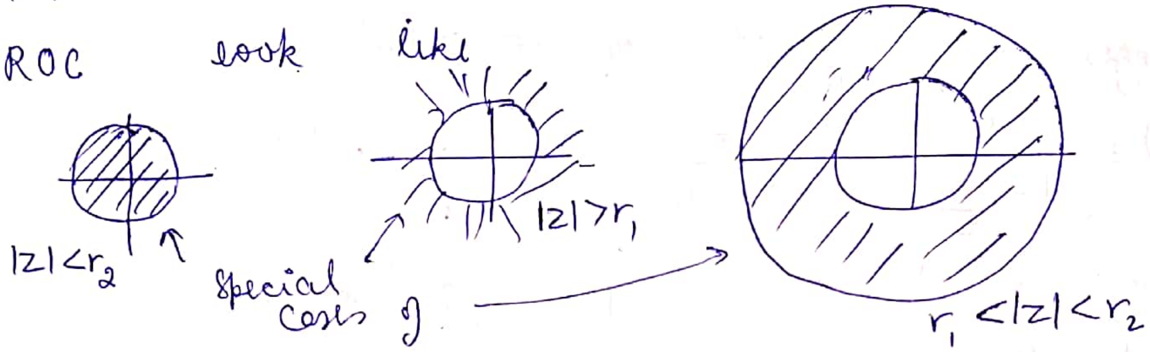
The ROC of the z transform of $u[n]$ is $|r|>1$



∵ my ROC always involves circle ( dep. on r )

→ Convergence of the Z-transform depends only on $|z| = r$

• ROC look like




• When ROC include the unit circle, then I talk of about Z-transform & Fourier Transform.

• If ROC includes $|z| = 1$ $(r = 1)$, then DTFT exists (converges)

This convergence of DTFT has a special name called 'stability'

"If the Z-transform of an Impulse Response h[n] for an LTI system converges on the unit circle (DTFT exists), then the system is stable

$$X(z) = \frac{N(z)}{D(z)}$$ ← Polynomials in z

$N(z) = 0 \Rightarrow X(z) = 0$ "Zeroes"

$D(z) = 0 \Rightarrow X(z) = \infty$ "Poles"

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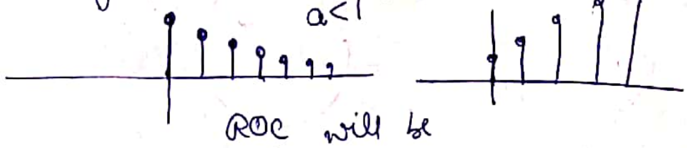
Ex. Right - sided exponential ( only starts at 0 goes to $\infty$)




$$x[n] = a^n u[n]$$

ROC will be for $a<1$

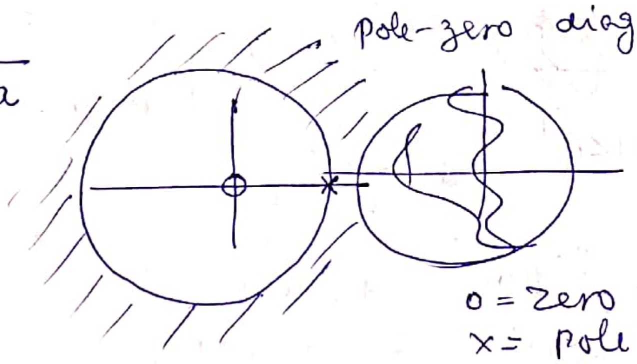
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

this converges if $\left|\frac{a}{z}\right| < 1$ or $|z| > |a|$

then $X(z) = \dfrac{1}{1-\frac{a}{z}} = \dfrac{z}{z-a}$

Pole-zero diag.



If $|a|<1$ then the ROC includes the unit circle & F.T exists, otherwise FT doesn't converge.

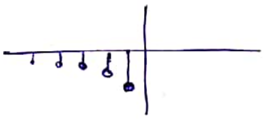
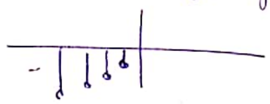
o = zero
x = pole
(considering a to be real +ve)

## Left - sided exponential

$$x[n] = -a^n u[-n-1]$$

↑ "on" from $-\infty$ to $-1$

$n=0$ $u[-1]=0$
$n=1$ $u[-2]=0$
$n=-1$ $u[0]=1$
$n=-2$ $u[1]=1$

 or 

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[n-1] z^{-n}$$

$$-\left(\frac{a}{z}\right)^{-\infty} + .. -\left(\frac{a}{z}\right)^{-3} + \left(-a/z\right)^{-2}$$

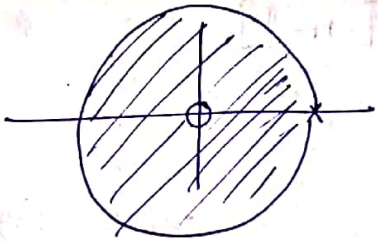
$$= \sum_{n=-\infty}^{-1} -\left(\frac{a}{z}\right)^n = \sum_{n=1}^{\infty} -\left(\frac{z}{a}\right)^n = -\sum_{n=0}^{\infty}\left(\frac{z}{a}\right)^n$$

The sum converges $\left|\frac{z}{a}\right| < 1$

$$= \frac{-1}{1-\frac{z}{a}} + 1$$

$$= \frac{-a}{a-z} + \frac{a-z}{a-z} = \frac{-z}{a-z} = \frac{z}{z-a}$$

## ROC



* Z-transform consists of both an algebraic form of $X(z)$ and an ROC that says where this is valid.
(use in Inverse Z-transform)

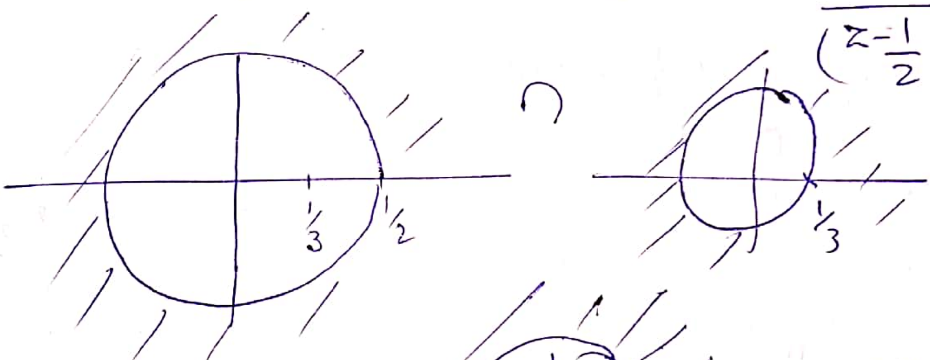
eg. (3)

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

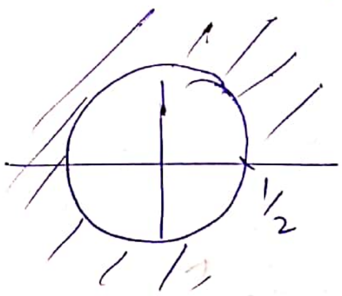
$$= \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}} = \frac{2z^2 - \frac{1}{6}z}{(z-\frac{1}{2})(z+\frac{1}{3})}$$

ROC $|z| > \frac{1}{2}$     ROC $|z| > \frac{1}{3}$

$$= \frac{2z\,(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$$

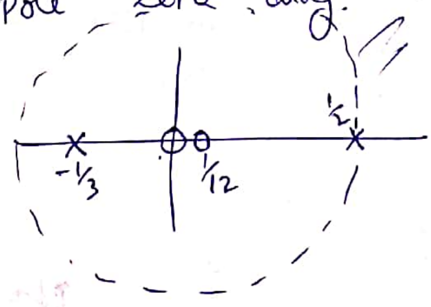



Final ROC




$|z| > \frac{1}{2}$

Pole zero diag.




ROC is related to the poles (~~Play~~ Poles play a huge role in what the ROC will be) → I need to look at things involved in the denominator.

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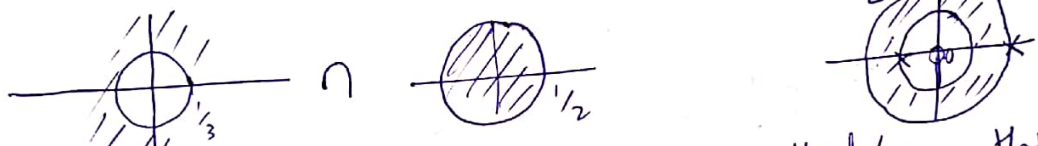
eg. $x[n] = \underbrace{\left(\frac{-1}{3}\right)^n u[n]}_{\text{Right sided}} - \underbrace{\left(\frac{1}{2}\right)^n u[-n-1]}_{\text{Left sided}}$

$$X(z) = \frac{\cancel{z(}\, 2z\left(z - \frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

same algebraic version but diff. ROC

ROC : $|z| > \frac{1}{3}$ $\cap$ ROC : $|z| < \frac{1}{2}$




∴ 3 possibilities for valid Z-transform that're the same pole-zero plot.

- most of the time the signals we'll be considering are impulse response & impulse resp. of system that're causal are Right ~~sided~~ sided signals

<u>Ex.</u> Finite-length exponential

$$x[n] = \begin{cases} a^n & n \in [0, N-1] \\ 0 & \text{else} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(\frac{a}{z}\right)^n$$

finite no. of terms

$$= \frac{1 - \left(\frac{a}{z}\right)^N}{1 - \frac{a}{z}}$$

finite sum formula is always valid

$$= \frac{z^N - a^N}{z^N - a z^{N-1}} = \frac{z^N - a^N}{z^{N-1}(z-a)}$$

N zeroes ~~(at z=0)~~
N poles
(N-1) poles at $z=0$
Pole at $z=a$

N zeroes at Roots $z^N = a^N$

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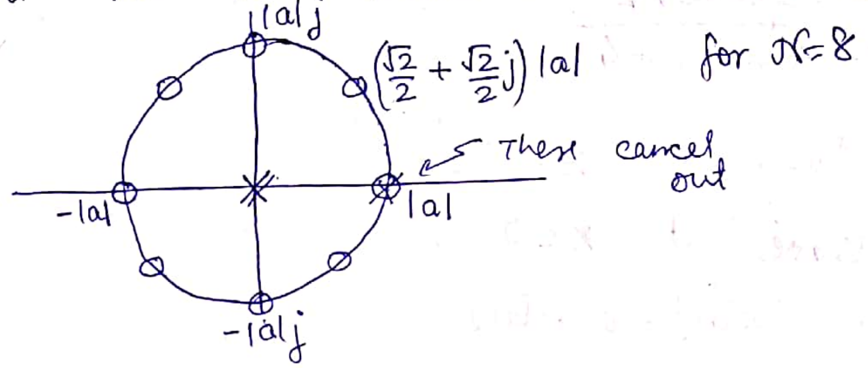
N - roots of a no. are distributed around a circle




ROC is $|z|>0$

$\therefore$ Any value of $z$ I can look at the Z-transform for the signal

- This is gen. true when impulse response is of finite - length

Finite length I/P $\Rightarrow$ ROC is whole Z-plane (except possibly $|z|=0$ )

eg. $x[n]=2^n \cos(3n)\, u[n]$

$$=2^n\left(\frac{e^{j3n}+e^{-j3n}}{2}\right)u[n]$$

$$=\frac{1}{2}\left(\left(2e^{3j}\right)^n+\left(2e^{-3j}\right)^n\right)\quad u[n]$$

$$X(z)=\frac{1}{2}\left(\frac{1}{1-\frac{2e^{3j}}{z}}\right)+\frac{1}{2}\left(\frac{1}{1-\frac{2e^{-3j}}{z}}\right)$$

Rem. $x[n]=a^n u[n]$ 'a' can be complex no.

$$X(z)=\cancel{\sum x[n] z^{-n}}\ \frac{1}{1-\frac{a}{z}}=\frac{z}{z-a}\qquad |a/z|<1$$

$$=\frac{1}{2}\cdot\frac{z}{z-2e^{3j}}+\frac{1}{2}\cdot\frac{z}{z-2e^{-3j}}$$

$$=\frac{1}{2}\,\frac{z(z-2e^{-3j})+z(z-2e^{3j})}{(z-2e^{3j})(z-2e^{-3j})}$$

$$=\frac{1}{2}\,\frac{2z^2-2z(e^{-3j}+e^{3j})}{z^2-2z(e^{-3j}+e^{3j})+4}=\frac{2z^2-(4\cos 3)z}{2(z^2-(4\cos 3)z+4)}$$

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$$\frac{z^2 - (2\cos 3) z}{z^2 - 4\cos 3 z + 4}$$

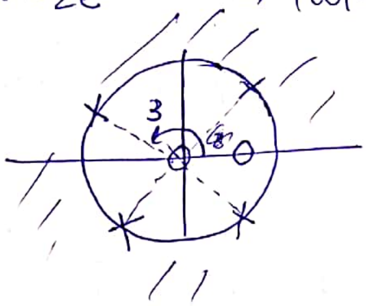
ROC:- Poles at $z = 2e^{\pm 3j}$

Zeroes at $z = 0$ ; $z = 2\cos 3$

ROC for $x[n] = a^n u[n]$

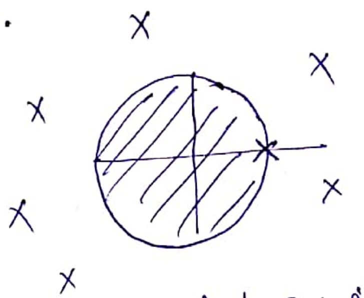
was $|z| > a$

$a = 2e^{\pm 3j} \rightarrow |a| = 2$ $\qquad |z| > 2$ $\qquad e^{\pm 3j}$ ✓ radian close to $\pi$

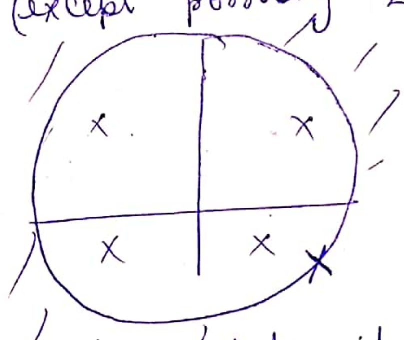



Rules about ROC :-

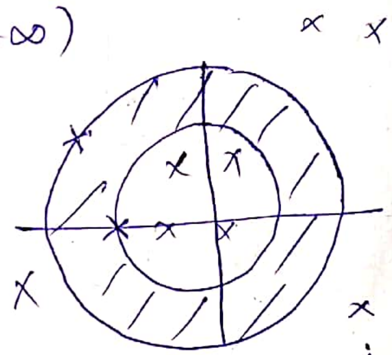
- ROC is a ring / disc centred at origin
- ROC contains no poles (Z-transform is $\infty$ → not converging)
- If $x[n]$ is finite length the ROC is the entire z plane (except possibly $z=0$, $z=\infty$)



Left sided signal
ROC is region inside the "inner most pole"



Right sided signal
ROC is region outside the "outermost pole"



Two-sided signal
"Dough-Nut"
ROC is bounded by the poles

- ROC tells us is the system stable ?
  ↳ for a Transfer funcn. $H(z)$ is the system causal?

Stable ⟺ ROC includes the unit circle

Causal ⟺ Impulse response is right sided

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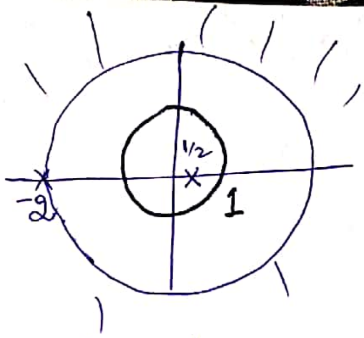
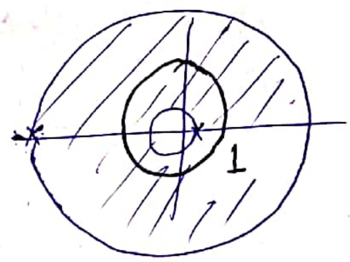
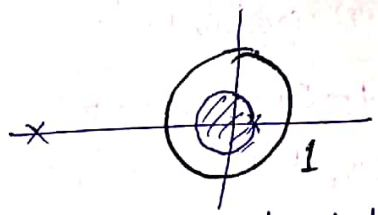



- Causal
- not stable



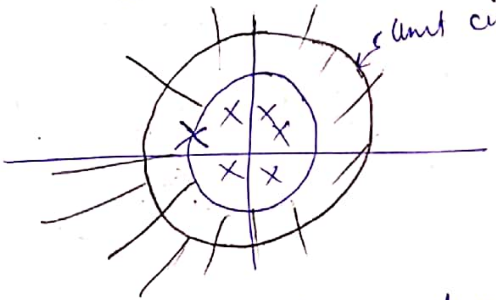

- Stable
- not causal




- not stable
- not causal

desire :- all the poles are inside the unit circle & the ROC looks like a star burst going outside outermost pole

- most desirable
  - ROC looks like




i.e ROC extends outwards from the largest magnitude pole & all poles are inside the unit circle

Root locus Routh Hurwitz can be applied here as well.

→ Inverse Z - Transform

$$\alpha^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha} \qquad |z| > |\alpha|$$

$$\cos\omega_0 n \, u[n] \overset{Z}{\longleftrightarrow} \frac{1-(\cos\omega_0)z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}} = \frac{z^2-(\cos\omega_0)z}{z^2-(2\cos\omega_0)z+1} \qquad |z|>1$$

$$r\sin\omega_0 n \, u[n] \overset{Z}{\longleftrightarrow} \frac{r\sin\omega_0 z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2 z^{-1}} = r\sin\omega \qquad |z| > r$$

say $x[n]=0 \quad n<0$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n} +$$

$$= x[0] + x[1] z^{-1} + x[2] z^{-2} + \dots$$

∴ for a causal signal only negative powers of z

$$\frac{1-(\cos\omega_0)z^{-1}}{1-2\cos\omega_0 z^{-1}+z^{-2}}\cdot\frac{z^2}{z^2}=\frac{z^2-(\cos\omega_0)z}{z^2-(2\cos\omega_0)z+1}$$

- Inverse Z → -ve powers of z

→ what is the inverse Z-transform

$$x[n]=\frac{1}{2\pi j}\oint X(z)\,z^{n-1}\,dz$$

complex contour integral for some $|z|=r$ in the ROC

eg.

$$X(z)=\frac{7-13z^{-1}}{1-2z^{-1}-3z^{-2}} \qquad |z|>1$$

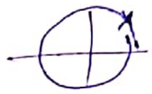
I. Factor the denominator

$(1-3z^{-1})(1+z^{-1})$

(consider $1-2z-3z^2$)
$1-3z+z-3z^2$
$(1-3z)+z(1-3z)$

or multiply by $z^2$

ROC can give clue.
(one of the poles is at $|z|=1$)



$$=\frac{A}{1-3z^{-1}}+\frac{B}{1+z^{-1}}$$

$$=\frac{(A+B)+(A-3B)z^{-1}}{(1-3z^{-1})(1+z^{-1})}$$

~~$y[0]=1$~~
~~$y[1]=\alpha$~~
~~$y[n]=\alpha(1+\alpha+\alpha^2+\dots\alpha^n)-(1+\alpha+\alpha^2\dots\alpha^{n-3}+\alpha^{n-2}+\alpha^{n-1}+\alpha^n)$~~

$A+B=7$
$A-3B=-13$

$4B=20$
$B=5$
$A=2$

$$=\frac{2}{1-3z^{-1}}+\frac{5}{1+z^{-1}}$$

$$=2\left(\frac{1}{3}\right)^n u[n]+5(-1)^n u[n]$$

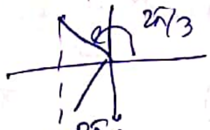
(3)

eg. $$X(z)=\frac{3z}{z^2+2z+4}$$ Right sided signal

$$=\frac{3z^{-1}}{1+2z^{-1}+4z^{-2}}$$

$$\frac{-2\pm\sqrt{4-16}}{2}$$

$$=-1\pm\sqrt{3}j$$



$2\pi/3$

$$=2e^{\pm\frac{2\pi}{3}j}$$

$r$   $\omega_0$

$r=2$   $2r\cos\omega_0=-2$   match
$2\cdot 2\cdot -\frac{1}{2}=-2$

eg. $X(z) = \frac{3z+5}{z^2+2z+4} = Ar^n \sin \omega_0 n \; u[n] + Br^n \cos\omega_0 n \; u[n]$

—

$X(z) = 3z^2 + 5z^{-1} - \frac{1}{2} + 3z^3$

ROC $0 < |z| < \infty$

what is def. of z Transform

basically a series in powers of z

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots$$
$$+ x[-1]z^{1} + x[-2]z^{2} + \dots$$

Coeff of z are actual values of x

$$x[n] = -\frac{1}{2}\delta[n] + 5\delta[n-1] + 3\delta[n-2] + 3\delta[n+3]$$

what is x[2]? coeff. on $z^{-2}$

• $X(z) = e^{z}$ $\quad |z| < \infty$

power series

$$= 1 + z + \frac{1}{2}z^2 + \frac{z^3}{3!} + \dots$$

$= \; x[0] \quad x[-1] \quad x[-2]$

Left sided signal

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) z^{n}$$

$$x[n] = \begin{cases} 0 & n \geq 1 \\ \frac{1}{(-n)!} & n \leq 0 \end{cases}$$

<u>Long division</u> :-

$$X(z) = \frac{1+2z^{-1}}{1+z^{-1}}$$

I don't have things where numerator has extra power of z.

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$$1+z^{-1} \overline{\big) 1+2z^{-1}} \big( 1+z^{-1}-z^{-2}+z^{-3}+\dots$$

$$\begin{array}{r} 1+z^{-1} \\ \hline z^{-1} \\ z^{-1}+z^{-2} \\ \hline -z^{-2} \\ -z^{-2} \\ \hline \end{array}$$

make a power series

$\boxed{z^{-\infty}=0}$

Properties:

1) Linearity
$$a x_1[n] + b x_2[n] \longleftrightarrow a X_1(z) + b X_2(z)$$

2) Time shift $\quad x[n-n_0] \longleftrightarrow z^{-n_0} X(z)$

In Fourier $e^{j\omega_0}$ may introduce extra poles or zeroes.

(Q) $$X(z) = \frac{1+2z^{-1}}{1+z^{-1}}$$

$$= \frac{1}{1+z^{-1}} + 2z^{-1}\cdot\frac{1}{1+z^{-1}}$$

$$= (-1)^n u[n] + 2(-1)^{n-1} u[n-1]$$

alternate pattern of +1s & -1s.

Degree of polynomial in denominator should be higher than degree of polynomial in the numerator. (If there're big powers, take them out & use as delays)

③ Scaling :-
$$a^n x[n] \longleftrightarrow X\left(\frac{z}{a}\right)$$

④ Time Reversal
$$x[-n] \longleftrightarrow X\left(\frac{1}{z}\right)$$

Inverse of poles & zeroes across unit circle

⑤ Convolution :
$$x[n] * h[n] \longleftrightarrow X(z)\cdot H(z)$$

⑥ Differentiation $\quad n x[n] \longleftrightarrow -z \dfrac{dX(z)}{dz}$

7. Initial value theorem:

$$x[0] = \lim_{z \to \infty} X(z)$$

systems where

$$H(z) = \frac{N(z)}{D(z)} \leftarrow \text{Polynomials}$$

Transfer funcn.

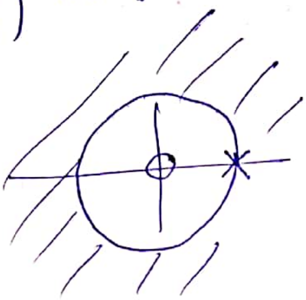
$N(z) = 0 \Rightarrow H(z) = 0 \Rightarrow$ Zeroes

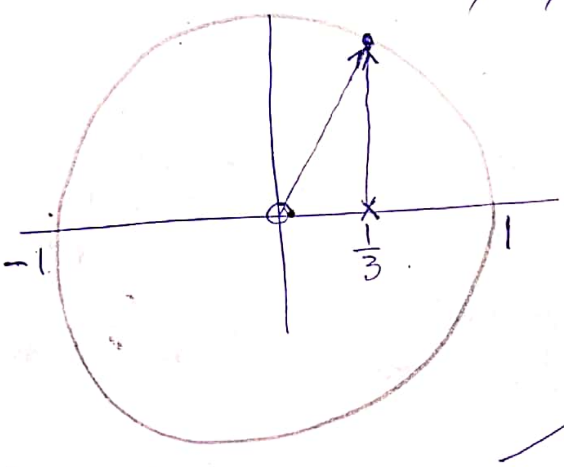
$D(z) = 0 \Rightarrow H(z) = \infty \Rightarrow$ Poles

Reading frequency Resp. from Z-Transform:

$$h[n] = \left(\frac{1}{3}\right)^n u[n] \qquad H(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} \qquad |z| > \frac{1}{3}$$

$$\frac{z}{z-3}$$

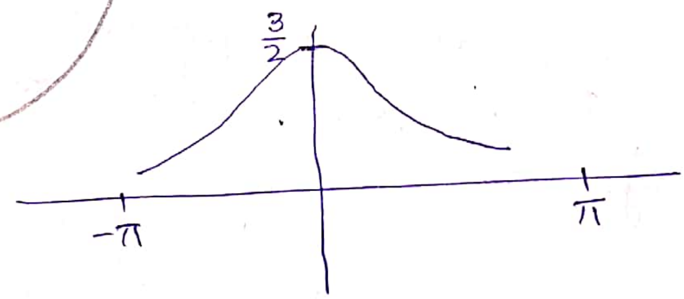





$$|H(e^{j\omega_0})| = \left| \frac{N(e^{j\omega_0})}{D(e^{j\omega_0})} \right| = \frac{\prod(\text{length of vector from each 0 to } e^{j\omega_0})}{\prod(\text{length of vector from each pole to } e^{j\omega_0})}$$

Product

$|H(\omega)|$




As we move around, the ratio gets smaller & smaller

How are the poles/zeroes affecting the position g'n on the unit circle

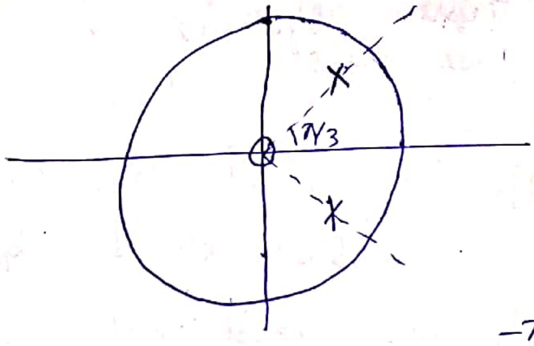
ex.

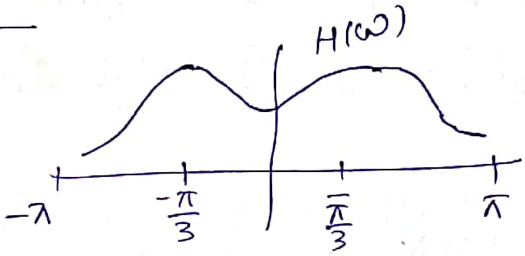
$$h[n] = \left(\frac{5}{6}\right)^n \sin\left(\frac{\pi}{3}n\right) u[n]$$

$$H(z) = \frac{\frac{5}{6}\sin\left(\frac{\pi}{3}\right) z^{-1}}{1 - \frac{10}{6}\cos\frac{\pi}{3} z^{-1} + \left(\frac{5}{6}\right)^2}$$

Scanned by CamScanner

where is the influence of these poles felt?

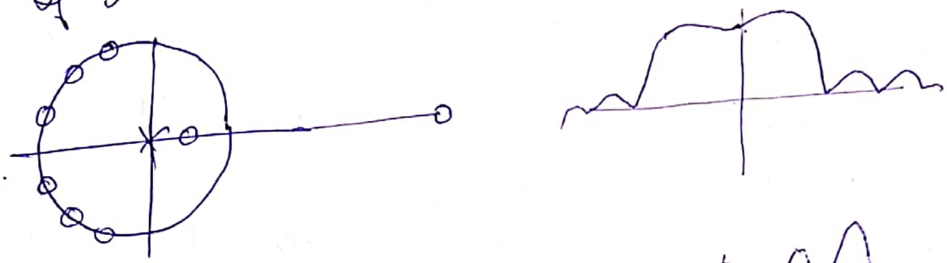


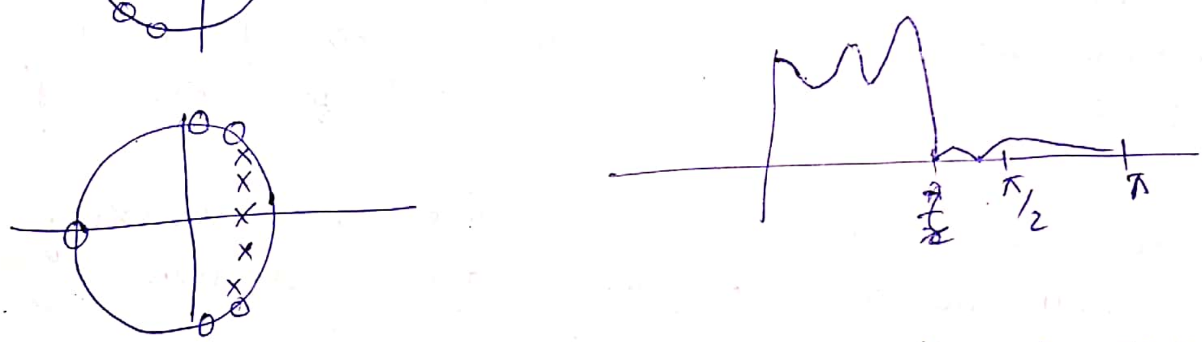




At origin the influence is still larger than at π

Crude Band-Pass Filter

• If I move the poles inwards.






How many poles and zeroes are given → In DSP called 'Taps'.

Z are used for solving difference eqns.

$$y[n] + \frac{1}{4} y[n-1] = x[n] + \frac{1}{5} x[n-1]$$

what is H(Z)

$$Y(z) + \frac{1}{4} z^{-1} Y(z) = X(z) + \frac{1}{5} z^{-1} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{5} z^{-1}}{1 + \frac{1}{4} z^{-1}}$$

$|z| > \frac{1}{4}$ for right-handed I. R

what is response to unit step?

$$X(z) = \frac{1}{1 - z^{-1}} \quad ; \quad Y(z) = \frac{1 + \frac{1}{5} z^{-1}}{(1 + \frac{1}{4} z^{-1})(1 - z^{-1})}$$

$$= \frac{A}{(1 + \frac{1}{4} z^{-1})} + \frac{B}{(1 - z^{-1})}$$