6.1 INTRODUCTION

So far we have limited our study to resistive circuits. In this chapter, we shall introduce two new and important passive linear circuit elements: the capacitor and the inductor. Unlike resistors, which dissipate energy, capacitors and inductors do not dissipate but store energy, which can be retrieved at a later time. For this reason, capacitors and inductors are called *storage* elements.

In contrast to a resistor, which spends or dissipates energy irreversibly, an inductor or capacitor stores or releases energy (i.e., has a memory).

The application of resistive circuits is quite limited. With the introduction of capacitors and inductors in this chapter, we will be able to analyze more important and practical circuits. Be assured that the circuit analysis techniques covered in Chapters 3 and 4 are equally applicable to circuits with capacitors and inductors.

We begin by introducing capacitors and describing how to combine them in series or in parallel. Later, we do the same for inductors. As typical applications, we explore how capacitors are combined with op amps to form integrators, differentiators, and analog computers.

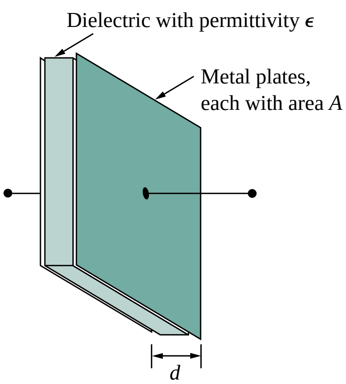

Figure 6.1 A typical capacitor.

6.2 CAPACITORS

A capacitor is a passive element designed to store energy in its electric field. Besides resistors, capacitors are the most common electrical components. Capacitors are used extensively in electronics, communications, computers, and power systems. For example, they are used in the tuning circuits of radio receivers and as dynamic memory elements in computer systems.

A capacitor is typically constructed as depicted in Fig. 6.1.

> A **capacitor** consists of two conducting plates separated by an insulator (or dielectric).

In many practical applications, the plates may be aluminum foil while the dielectric may be air, ceramic, paper, or mica.

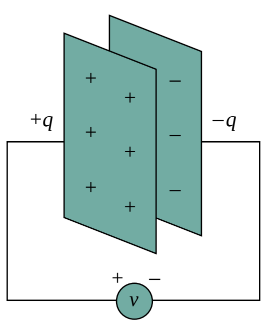

Figure 6.2 A capacitor with applied voltage v.

When a voltage source v is connected to the capacitor, as in Fig. 6.2, the source deposits a positive charge q on one plate and a negative charge $-q$ on the other. The capacitor is said to store the electric charge. The amount of charge stored, represented by q, is directly proportional to the applied voltage v so that

$$q = Cv \tag{6.1}$$

where C, the constant of proportionality, is known as the *capacitance* of the capacitor. The unit of capacitance is the farad (F), in honor of the English physicist Michael Faraday (1791–1867). From Eq. (6.1), we may derive the following definition.

Alternatively, capacitance is the amount of charge stored per plate for a unit voltage difference in a capacitor.

> **Capacitance** is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates, measured in farads (F).

Note from Eq. (6.1) that 1 farad = 1 coulomb/volt.

Although the capacitance C of a capacitor is the ratio of the charge q per plate to the applied voltage v, it does not depend on q or v. It depends on the physical dimensions of the capacitor. For example, for the parallel-plate capacitor shown in Fig. 6.1, the capacitance is given by

$$C = \frac{\epsilon A}{d} \tag{6.2}$$

Capacitor voltage rating and capacitance are typically inversely rated due to the relationships in Eqs. (6.1) and (6.2). Arcing occurs if d is small and V is high.

where A is the surface area of each plate, d is the distance between the plates, and ϵ is the permittivity of the dielectric material between the plates. Although Eq. (6.2) applies to only parallel-plate capacitors, we may infer from it that, in general, three factors determine the value of the capacitance:

1. The surface area of the plates—the larger the area, the greater the capacitance.
2. The spacing between the plates—the smaller the spacing, the greater the capacitance.
3. The permittivity of the material—the higher the permittivity, the greater the capacitance.

Capacitors are commercially available in different values and types. Typically, capacitors have values in the picofarad (pF) to microfarad (μF) range. They are described by the dielectric material they are made of and by whether they are of fixed or variable type. Figure 6.3 shows the circuit symbols for fixed and variable capacitors. Note that according to the passive sign convention, current is considered to flow into the positive terminal of the capacitor when the capacitor is being charged, and out of the positive terminal when the capacitor is discharging.

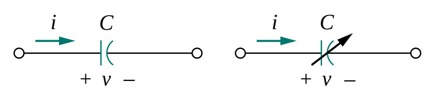

Figure 6.3 Circuit symbols for capacitors: (a) fixed capacitor, (b) variable capacitor.

Figure 6.4 shows common types of fixed-value capacitors. Polyester capacitors are light in weight, stable, and their change with temperature is predictable. Instead of polyester, other dielectric materials such as mica and polystyrene may be used. Film capacitors are rolled and housed in metal or plastic films. Electrolytic capacitors produce very high capacitance. Figure 6.5 shows the most common types of variable capacitors. The capacitance of a trimmer (or padder) capacitor or a glass piston capacitor is varied by turning the screw. The trimmer capacitor is often placed in parallel with another capacitor so that the equivalent capacitance can be varied slightly. The capacitance of the variable air capacitor (meshed plates) is varied by turning the shaft. Variable capacitors are used in radio

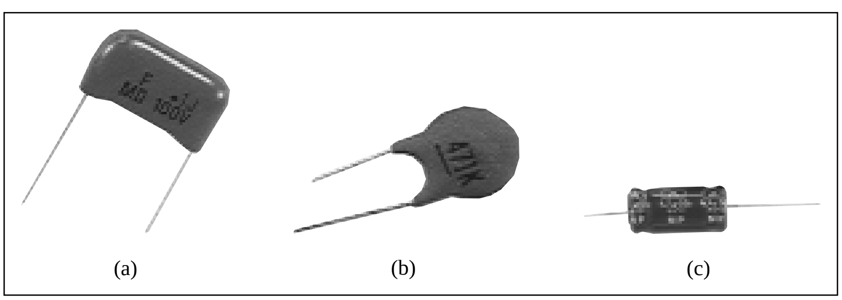

Figure 6.4 Fixed capacitors: (a) polyester capacitor, (b) ceramic capacitor, (c) electrolytic capacitor. *(Courtesy of Tech America.)*

receivers allowing one to tune to various stations. In addition, capacitors are used to block dc, pass ac, shift phase, store energy, start motors, and suppress noise.

To obtain the current-voltage relationship of the capacitor, we take the derivative of both sides of Eq. (6.1). Since

$$i = \frac{dq}{dt} \tag{6.3}$$

differentiating both sides of Eq. (6.1) gives

$$i = C\frac{dv}{dt} \tag{6.4}$$

This is the current-voltage relationship for a capacitor, assuming the positive sign convention. The relationship is illustrated in Fig. 6.6 for a capacitor whose capacitance is independent of voltage. Capacitors that satisfy Eq. (6.4) are said to be *linear.* For a *nonlinear capacitor*, the plot of the current-voltage relationship is not a straight line. Although some capacitors are nonlinear, most are linear. We will assume linear capacitors in this book.

The voltage-current relation of the capacitor can be obtained by integrating both sides of Eq. (6.4). We get

$$v = \frac{1}{C}\int_{-\infty}^{t} i\, dt \tag{6.5}$$

or

$$v = \frac{1}{C}\int_{t_0}^{t} i\, dt + v(t_0) \tag{6.6}$$

where $v(t_0) = q(t_0)/C$ is the voltage across the capacitor at time t_0. Equation (6.6) shows that capacitor voltage depends on the past history of the capacitor current. Hence, the capacitor has memory—a property that is often exploited.

The instantaneous power delivered to the capacitor is

$$p = vi = Cv\frac{dv}{dt} \tag{6.7}$$

The energy stored in the capacitor is therefore

$$w = \int_{-\infty}^{t} p\, dt = C\int_{-\infty}^{t} v\frac{dv}{dt}dt = C\int_{-\infty}^{t} v\, dv = \frac{1}{2}Cv^2\Big|_{t=-\infty}^{t} \tag{6.8}$$

We note that $v(-\infty) = 0$, because the capacitor was uncharged at $t = -\infty$. Thus,

$$w = \frac{1}{2}Cv^2 \tag{6.9}$$

Using Eq. (6.1), we may rewrite Eq. (6.9) as

$$w = \frac{q^2}{2C} \tag{6.10}$$

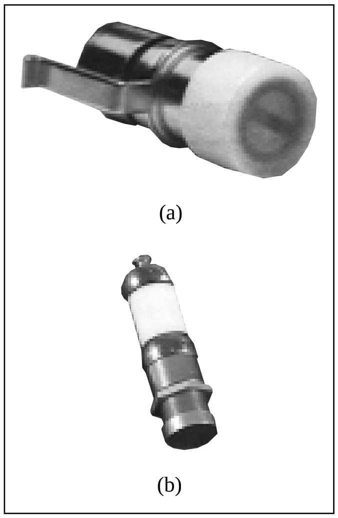

Figure 6.5 Variable capacitors: (a) trimmer capacitor, (b) filmtrim capacitor.
(Courtesy of Johanson.)

According to Eq. (6.4), for a capacitor to carry current, its voltage must vary with time. Hence, for constant voltage, $i = 0$.

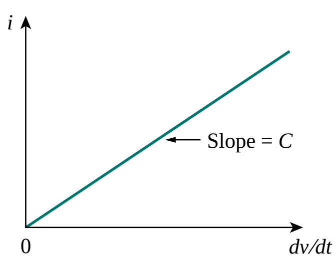

Figure 6.6 Current-voltage relationship of a capacitor.

Equation (6.9) or (6.10) represents the energy stored in the electric field that exists between the plates of the capacitor. This energy can be retrieved, since an ideal capacitor cannot dissipate energy. In fact, the word *capacitor* is derived from this element's capacity to store energy in an electric field.

We should note the following important properties of a capacitor:

1. Note from Eq. (6.4) that when the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero. Thus,

> A capacitor is an open circuit to dc.

However, if a battery (dc voltage) is connected across a capacitor, the capacitor charges.

2. The voltage on the capacitor must be continuous.

> The voltage on a capacitor cannot change abruptly.

The capacitor resists an abrupt change in the voltage across it. According to Eq. (6.4), a discontinuous change in voltage requires an infinite current, which is physically impossible. For example, the voltage across a capacitor may take the form shown in Fig. 6.7(a), whereas it is not physically possible for the capacitor voltage to take the form shown in Fig. 6.7(b) because of the abrupt change. Conversely, the current through a capacitor can change instantaneously.

3. The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit.
4. A real, nonideal capacitor has a parallel-model leakage resistance, as shown in Fig. 6.8. The leakage resistance may be as high as 100 MΩ and can be neglected for most practical applications. For this reason, we will assume ideal capacitors in this book.

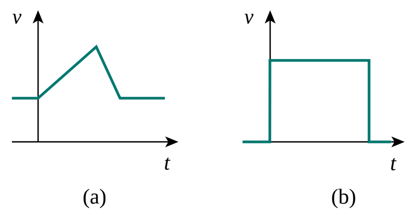

Figure 6.7 Voltage across a capacitor: (a) allowed, (b) not allowable; an abrupt change is not possible.

An alternative way of looking at this is using Eq. (6.9), which indicates that energy is proportional to voltage squared. Since injecting or extracting energy can only be done over some finite time, voltage cannot change instantaneously across a capacitor.

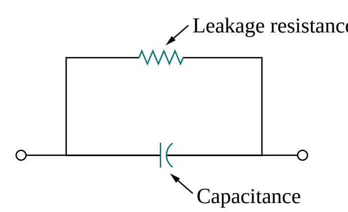

Figure 6.8 Circuit model of a nonideal capacitor.

EXAMPLE 6.1

(a) Calculate the charge stored on a 3-pF capacitor with 20 V across it.
(b) Find the energy stored in the capacitor.

Solution:

(a) Since $q = Cv$,

$$q = 3 \times 10^{-12} \times 20 = 60 \text{ pC}$$

(b) The energy stored is

$$w = \frac{1}{2}Cv^2 = \frac{1}{2} \times 3 \times 10^{-12} \times 400 = 600 \text{ pJ}$$

Solution:

We first find the equivalent capacitance C_{eq}, shown in Fig. 6.19. The two parallel capacitors in Fig. 6.18 can be combined to get $40 + 20 = 60$ mF. This 60-mF capacitor is in series with the 20-mF and 30-mF capacitors. Thus,

$$C_{eq} = \frac{1}{\frac{1}{60} + \frac{1}{30} + \frac{1}{20}} \text{ mF} = 10 \text{ mF}$$

The total charge is

$$q = C_{eq}v = 10 \times 10^{-3} \times 30 = 0.3 \text{ C}$$

This is the charge on the 20-mF and 30-mF capacitors, because they are in series with the 30-V source. (A crude way to see this is to imagine that charge acts like current, since $i = dq/dt$.) Therefore,

$$v_1 = \frac{q}{C_1} = \frac{0.3}{20 \times 10^{-3}} = 15 \text{ V} \qquad v_2 = \frac{q}{C_2} = \frac{0.3}{30 \times 10^{-3}} = 10 \text{ V}$$

Having determined v_1 and v_2, we now use KVL to determine v_3 by

$$v_3 = 30 - v_1 - v_2 = 5 \text{ V}$$

Alternatively, since the 40-mF and 20-mF capacitors are in parallel, they have the same voltage v_3 and their combined capacitance is $40 + 20 = 60$ mF. This combined capacitance is in series with the 20-mF and 30-mF capacitors and consequently has the same charge on it. Hence,

$$v_3 = \frac{q}{60 \text{ mF}} = \frac{0.3}{60 \times 10^{-3}} = 5 \text{ V}$$

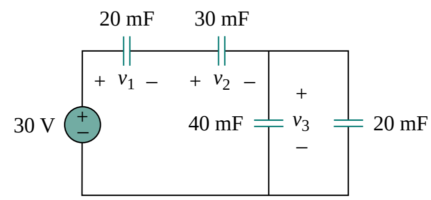

Figure 6.18 For Example 6.7.

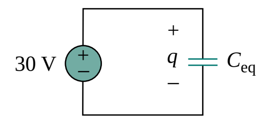

Figure 6.19 Equivalent circuit for Fig. 6.18.

PRACTICE PROBLEM 6.7

Find the voltage across each of the capacitors in Fig. 6.20.

Answer: $v_1 = 30$ V, $v_2 = 30$ V, $v_3 = 10$ V, $v_4 = 20$ V.

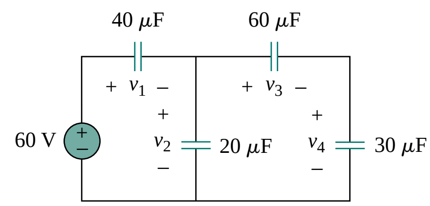

Figure 6.20 For Practice Prob. 6.7.

6.4 INDUCTORS

An inductor is a passive element designed to store energy in its magnetic field. Inductors find numerous applications in electronic and power systems. They are used in power supplies, transformers, radios, TVs, radars, and electric motors.

Any conductor of electric current has inductive properties and may be regarded as an inductor. But in order to enhance the inductive effect, a practical inductor is usually formed into a cylindrical coil with many turns of conducting wire, as shown in Fig. 6.21.

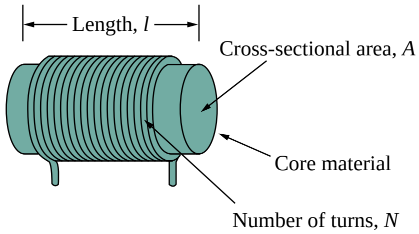

Figure 6.21 Typical form of an inductor.

An inductor consists of a coil of conducting wire.

If current is allowed to pass through an inductor, it is found that the voltage across the inductor is directly proportional to the time rate of change of the current. Using the passive sign convention,

$$v = L\frac{di}{dt} \tag{6.18}$$

where L is the constant of proportionality called the *inductance* of the inductor. The unit of inductance is the henry (H), named in honor of the American inventor Joseph Henry (1797–1878). It is clear from Eq. (6.18) that 1 henry equals 1 volt-second per ampere.

In view of Eq. (6.18), for an inductor to have voltage across its terminals, its current must vary with time. Hence, $v = 0$ for constant current through the inductor.

Inductance is the property whereby an inductor exhibits opposition to the change of current flowing through it, measured in henrys (H).

The inductance of an inductor depends on its physical dimension and construction. Formulas for calculating the inductance of inductors of different shapes are derived from electromagnetic theory and can be found in standard electrical engineering handbooks. For example, for the inductor (solenoid) shown in Fig. 6.21,

$$L = \frac{N^2 \mu A}{\ell} \tag{6.19}$$

where N is the number of turns, ℓ is the length, A is the cross-sectional area, and μ is the permeability of the core. We can see from Eq. (6.19) that inductance can be increased by increasing the number of turns of coil, using material with higher permeability as the core, increasing the cross-sectional area, or reducing the length of the coil.

Like capacitors, commercially available inductors come in different values and types. Typical practical inductors have inductance values ranging from a few microhenrys (μH), as in communication systems, to tens of henrys (H) as in power systems. Inductors may be fixed or variable. The core may be made of iron, steel, plastic, or air. The terms *coil* and *choke* are also used for inductors. Common inductors are shown in Fig. 6.22. The circuit symbols for inductors are shown in Fig. 6.23, following the passive sign convention.

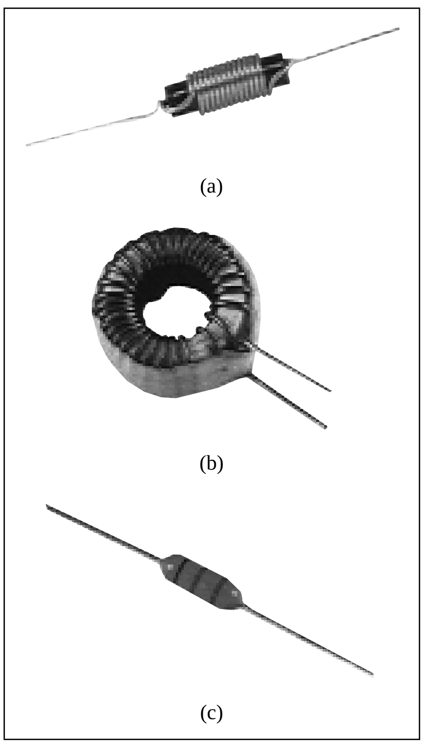

Figure 6.22 Various types of inductors: (a) solenoidal wound inductor, (b) toroidal inductor, (c) chip inductor. *(Courtesy of Tech America.)*

Equation (6.18) is the voltage-current relationship for an inductor. Figure 6.24 shows this relationship graphically for an inductor whose inductance is independent of current. Such an inductor is known as a *linear inductor*. For a *nonlinear inductor*, the plot of Eq. (6.18) will not be a straight line because its inductance varies with current. We will assume linear inductors in this textbook unless stated otherwise.

The current-voltage relationship is obtained from Eq. (6.18) as

$$di = \frac{1}{L} v\, dt$$

Integrating gives

$$i = \frac{1}{L}\int_{-\infty}^{t} v(t)\,dt \tag{6.20}$$

or

$$i = \frac{1}{L}\int_{t_0}^{t} v(t)\,dt + i(t_0) \tag{6.21}$$

where $i(t_0)$ is the total current for $-\infty < t < t_0$ and $i(-\infty) = 0$. The idea of making $i(-\infty) = 0$ is practical and reasonable, because there must be a time in the past when there was no current in the inductor.

The inductor is designed to store energy in its magnetic field. The energy stored can be obtained from Eqs. (6.18) and (6.20). The power delivered to the inductor is

$$p = vi = \left(L\frac{di}{dt}\right)i \tag{6.22}$$

The energy stored is

$$\begin{aligned} w &= \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t}\left(L\frac{di}{dt}\right)i\,dt \\ &= L\int_{-\infty}^{t} i\,di = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(-\infty) \end{aligned} \tag{6.23}$$

Since $i(-\infty) = 0$,

$$w = \frac{1}{2}Li^2 \tag{6.24}$$

We should note the following important properties of an inductor.

1. Note from Eq. (6.18) that the voltage across an inductor is zero when the current is constant. Thus,

> An inductor acts like a short circuit to dc.

2. An important property of the inductor is its opposition to the change in current flowing through it.

> The current through an inductor cannot change instantaneously.

According to Eq. (6.18), a discontinuous change in the current through an inductor requires an infinite voltage, which is not physically possible. Thus, an inductor opposes an abrupt change in the current through it. For example, the current through an inductor may take the form shown in Fig. 6.25(a), whereas the inductor current cannot take the form shown in Fig. 6.25(b) in real-life situations due to the discontinuities. However, the voltage across an inductor can change abruptly.

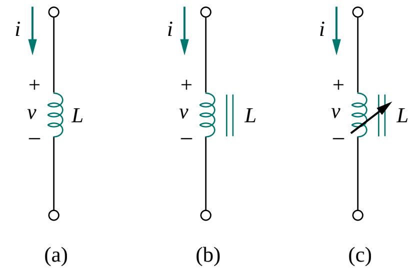

Figure 6.23 Circuit symbols for inductors: (a) air-core, (b) iron-core, (c) variable iron-core.

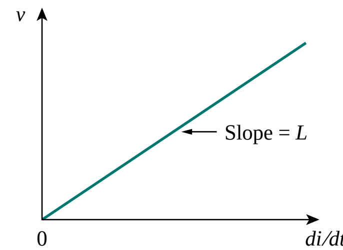

Figure 6.24 Voltage-current relationship of an inductor.

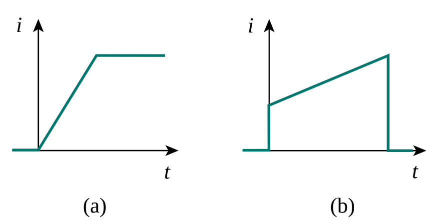

Figure 6.25 Current through an inductor: (a) allowed, (b) not allowable; an abrupt change is not possible.

3. Like the ideal capacitor, the ideal inductor does not dissipate energy. The energy stored in it can be retrieved at a later time. The inductor takes power from the circuit when storing energy and delivers power to the circuit when returning previously stored energy.

Since an inductor is often made of a highly conducting wire, it has a very small resistance.

4. A practical, nonideal inductor has a significant resistive component, as shown in Fig. 6.26. This is due to the fact that the inductor is made of a conducting material such as copper, which has some resistance. This resistance is called the *winding resistance* R_w, and it appears in series with the inductance of the inductor. The presence of R_w makes it both an energy storage device and an energy dissipation device. Since R_w is usually very small, it is ignored in most cases. The nonideal inductor also has a *winding capacitance* C_w due to the capacitive coupling between the conducting coils. C_w is very small and can be ignored in most cases, except at high frequencies. We will assume ideal inductors in this book.

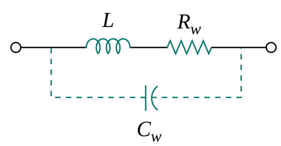

Figure 6.26 Circuit model for a practical inductor.

EXAMPLE 6.8

The current through a 0.1-H inductor is $i(t) = 10te^{-5t}$ A. Find the voltage across the inductor and the energy stored in it.

Solution:

Since $v = L\,di/dt$ and $L = 0.1$ H,

$$v = 0.1\frac{d}{dt}(10te^{-5t}) = e^{-5t} + t(-5)e^{-5t} = e^{-5t}(1-5t) \text{ V}$$

The energy stored is

$$w = \frac{1}{2}Li^2 = \frac{1}{2}(0.1)100t^2e^{-10t} = 5t^2e^{-10t} \text{ J}$$

PRACTICE PROBLEM 6.8

If the current through a 1-mH inductor is $i(t) = 20\cos 100t$ mA, find the terminal voltage and the energy stored.

Answer: $-2\sin 100t$ mV, $0.2\cos^2 100t$ μJ.

EXAMPLE 6.9

Find the current through a 5-H inductor if the voltage across it is

$$v(t) = \begin{cases} 30t^2, & t > 0 \\ 0, & t < 0 \end{cases}$$

Also find the energy stored within $0 < t < 5$ s.

Solution:

Since $i = \dfrac{1}{L}\displaystyle\int_{t_0}^{t} v(t)\,dt + i(t_0)$ and $L = 5$ H,

$$i = \frac{1}{5}\int_0^t 30t^2\,dt + 0 = 6 \times \frac{t^3}{3} = 2t^3 \text{ A}$$

11.1 INTRODUCTION

Our effort in ac circuit analysis so far has been focused mainly on calculating voltage and current. Our major concern in this chapter is power analysis.

Power analysis is of paramount importance. Power is the most important quantity in electric utilities, electronic, and communication systems, because such systems involve transmission of power from one point to another. Also, every industrial and household electrical device—every fan, motor, lamp, pressing iron, TV, personal computer—has a power rating that indicates how much power the equipment requires; exceeding the power rating can do permanent damage to an appliance. The most common form of electric power is 50- or 60-Hz ac power. The choice of ac over dc allowed high-voltage power transmission from the power generating plant to the consumer.

We will begin by defining and deriving *instantaneous power* and *average power*. We will then introduce other power concepts. As practical applications of these concepts, we will discuss how power is measured and reconsider how electric utility companies charge their customers.

11.2 INSTANTANEOUS AND AVERAGE POWER

As mentioned in Chapter 2, the *instantaneous power* $p(t)$ absorbed by an element is the product of the instantaneous voltage $v(t)$ across the element and the instantaneous current $i(t)$ through it. Assuming the passive sign convention,

$$\boxed{p(t) = v(t)i(t)} \tag{11.1}$$

The instantaneous power is the power at any instant of time. It is the rate at which an element absorbs energy.

We can also think of the instantaneous power as the power absorbed by the element at a specific instant of time. Instantaneous quantities are denoted by lowercase letters.

Consider the general case of instantaneous power absorbed by an arbitrary combination of circuit elements under sinusoidal excitation, as shown in Fig. 11.1. Let the voltage and current at the terminals of the circuit be

$$v(t) = V_m \cos(\omega t + \theta_v) \tag{11.2a}$$

$$i(t) = I_m \cos(\omega t + \theta_i) \tag{11.2b}$$

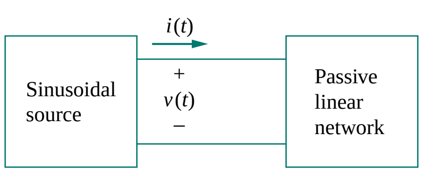

Figure 11.1 Sinusoidal source and passive linear circuit.

where V_m and I_m are the amplitudes (or peak values), and θ_v and θ_i are the phase angles of the voltage and current, respectively. The instantaneous power absorbed by the circuit is

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i) \tag{11.3}$$

We apply the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)] \tag{11.4}$$

and express Eq. (11.3) as

$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i) \tag{11.5}$$

This shows us that the instantaneous power has two parts. The first part is constant or time independent. Its value depends on the phase difference between the voltage and the current. The second part is a sinusoidal function whose frequency is 2ω, which is twice the angular frequency of the voltage or current.

A sketch of $p(t)$ in Eq. (11.5) is shown in Fig. 11.2, where $T = 2\pi/\omega$ is the period of voltage or current. We observe that $p(t)$ is periodic, $p(t) = p(t + T_0)$, and has a period of $T_0 = T/2$, since its frequency is twice that of voltage or current. We also observe that $p(t)$ is positive for some part of each cycle and negative for the rest of the cycle. When $p(t)$ is positive, power is absorbed by the circuit. When $p(t)$ is negative, power is absorbed by the source; that is, power is transferred from the circuit to the source. This is possible because of the storage elements (capacitors and inductors) in the circuit.

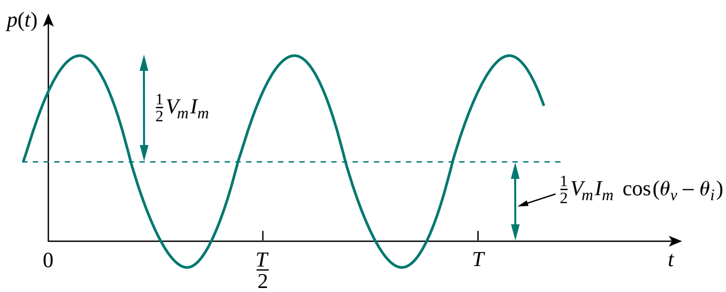

Figure 11.2 The instantaneous power $p(t)$ entering a circuit.

The instantaneous power changes with time and is therefore difficult to measure. The *average* power is more convenient to measure. In fact, the wattmeter, the instrument for measuring power, responds to average power.

The average power is the average of the instantaneous power over one period.

Thus, the average power is given by

$$P = \frac{1}{T}\int_0^T p(t)\,dt \tag{11.6}$$

Although Eq. (11.6) shows the averaging done over T, we would get the same result if we performed the integration over the actual period of $p(t)$ which is $T_0 = T/2$.

Substituting $p(t)$ in Eq. (11.5) into Eq. (11.6) gives

$$P = \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\,dt + \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)\,dt$$

$$= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \frac{1}{T} \int_0^T dt + \frac{1}{2} V_m I_m \frac{1}{T} \int_0^T \cos(2\omega t + \theta_v + \theta_i)\, dt \qquad (11.7)$$

The first integrand is constant, and the average of a constant is the same constant. The second integrand is a sinusoid. We know that the average of a sinusoid over its period is zero because the area under the sinusoid during a positive half-cycle is canceled by the area under it during the following negative half-cycle. Thus, the second term in Eq. (11.7) vanishes and the average power becomes

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \qquad (11.8)$$

Since $\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$, what is important is the difference in the phases of the voltage and current.

Note that $p(t)$ is time-varying while P does not depend on time. To find the instantaneous power, we must necessarily have $v(t)$ and $i(t)$ in the time domain. But we can find the average power when voltage and current are expressed in the time domain, as in Eq. (11.2), or when they are expressed in the frequency domain. The phasor forms of $v(t)$ and $i(t)$ in Eq. (11.2) are $\mathbf{V} = V_m \underline{/\theta_v}$ and $\mathbf{I} = I_m \underline{/\theta_i}$, respectively. P is calculated using Eq. (11.8) or using phasors $\mathbf{V}$ and $\mathbf{I}$. To use phasors, we notice that

$$\begin{aligned} \frac{1}{2}\mathbf{V}\mathbf{I}^* &= \frac{1}{2} V_m I_m \underline{/\theta_v - \theta_i} \\ &= \frac{1}{2} V_m I_m [\cos(\theta_v - \theta_i) + j \sin(\theta_v - \theta_i)] \end{aligned} \qquad (11.9)$$

We recognize the real part of this expression as the average power P according to Eq. (11.8). Thus,

$$\boxed{P = \frac{1}{2} \operatorname{Re}\left[\mathbf{V}\mathbf{I}^*\right] = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)} \qquad (11.10)$$

Consider two special cases of Eq. (11.10). When $\theta_v = \theta_i$, the voltage and current are in phase. This implies a purely resistive circuit or resistive load R, and

$$P = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R = \frac{1}{2} |\mathbf{I}|^2 R \qquad (11.11)$$

where $|\mathbf{I}|^2 = \mathbf{I} \times \mathbf{I}^*$. Equation (11.11) shows that a purely resistive circuit absorbs power at all times. When $\theta_v - \theta_i = \pm 90^\circ$, we have a purely reactive circuit, and

$$P = \frac{1}{2} V_m I_m \cos 90^\circ = 0 \qquad (11.12)$$

showing that a purely reactive circuit absorbs no average power. In summary,

> A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power.

EXAMPLE 11.1

Given that

$$v(t) = 120\cos(377t + 45^\circ)\ \text{V} \qquad \text{and} \qquad i(t) = 10\cos(377t - 10^\circ)\ \text{A}$$

find the instantaneous power and the average power absorbed by the passive linear network of Fig. 11.1.

Solution:

The instantaneous power is given by

$$p = vi = 1200\cos(377t + 45^\circ)\cos(377t - 10^\circ)$$

Applying the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

gives

$$p = 600[\cos(754t + 35^\circ) + \cos 55^\circ]$$

or

$$p(t) = 344.2 + 600\cos(754t + 35^\circ)\ \text{W}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}120(10)\cos[45^\circ - (-10^\circ)]$$
$$= 600\cos 55^\circ = 344.2\ \text{W}$$

which is the constant part of $p(t)$ above.

PRACTICE PROBLEM 11.1

Calculate the instantaneous power and average power absorbed by the passive linear network of Fig. 11.1 if

$$v(t) = 80\cos(10t + 20^\circ)\ \text{V} \qquad \text{and} \qquad i(t) = 15\sin(10t + 60^\circ)\ \text{A}$$

Answer: $385.7 + 600\cos(20t - 10^\circ)$ W, 385.7 W.

EXAMPLE 11.2

Calculate the average power absorbed by an impedance $\mathbf{Z} = 30 - j70\ \Omega$ when a voltage $\mathbf{V} = 120\underline{/0^\circ}$ is applied across it.

Solution:

The current through the impedance is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{120\underline{/0^\circ}}{30 - j70} = \frac{120\underline{/0^\circ}}{76.16\underline{/-66.8^\circ}} = 1.576\underline{/66.8^\circ} \text{ A}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}(120)(1.576)\cos(0 - 66.8^\circ) = 37.24 \text{ W}$$

PRACTICE PROBLEM 11.2

A current $\mathbf{I} = 10\underline{/30^\circ}$ flows through an impedance $\mathbf{Z} = 20\underline{/-22^\circ}\ \Omega$. Find the average power delivered to the impedance.

Answer: 927.2 W.

EXAMPLE 11.3

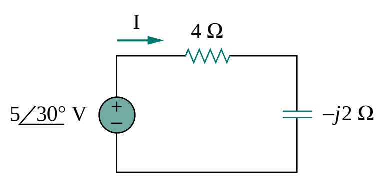

Figure 11.3 For Example 11.3.

For the circuit shown in Fig. 11.3, find the average power supplied by the source and the average power absorbed by the resistor.

Solution:

The current $\mathbf{I}$ is given by

$$\mathbf{I} = \frac{5\underline{/30^\circ}}{4 - j2} = \frac{5\underline{/30^\circ}}{4.472\underline{/-26.57^\circ}} = 1.118\underline{/56.57^\circ} \text{ A}$$

The average power supplied by the voltage source is

$$P = \frac{1}{2}(5)(1.118)\cos(30^\circ - 56.57^\circ) = 2.5 \text{ W}$$

The current through the resistor is

$$\mathbf{I} = \mathbf{I}_R = 1.118\underline{/56.57^\circ} \text{ A}$$

and the voltage across it is

$$\mathbf{V}_R = 4\mathbf{I}_R = 4.472\underline{/56.57^\circ} \text{ V}$$

The average power absorbed by the resistor is

$$P = \frac{1}{2}(4.472)(1.118) = 2.5 \text{ W}$$

which is the same as the average power supplied. Zero average power is absorbed by the capacitor.

PRACTICE PROBLEM 11.3

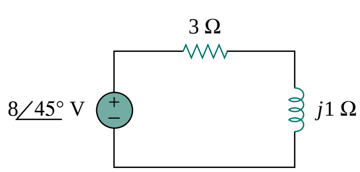

Figure 11.4 For Practice Prob. 11.3.

In the circuit of Fig. 11.4, calculate the average power absorbed by the resistor and inductor. Find the average power supplied by the voltage source.

Answer: 9.6 W, 0 W, 9.6 W.

EXAMPLE 11.4

Determine the power generated by each source and the average power absorbed by each passive element in the circuit of Fig. 11.5(a).

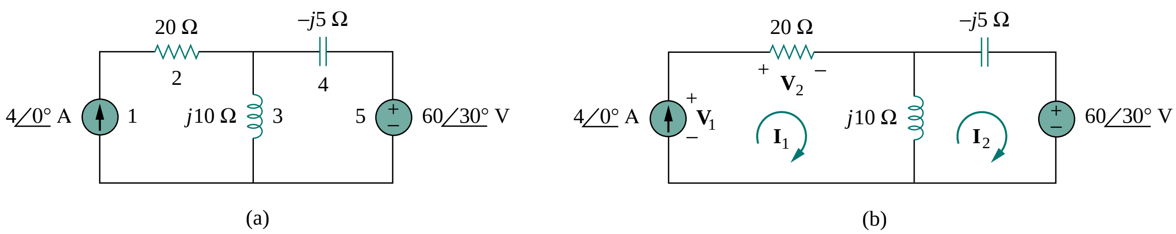

Figure 11.5 For Example 11.4.

Solution:

We apply mesh analysis as shown in Fig. 11.5(b). For mesh 1,

$$\mathbf{I}_1 = 4 \text{ A}$$

For mesh 2,

$$(j10 - j5)\mathbf{I}_2 - j10\mathbf{I}_1 + 60\angle 30^\circ = 0, \qquad \mathbf{I}_1 = 4 \text{ A}$$

or

$$j5\mathbf{I}_2 = -60\angle 30^\circ + j40 \quad \Longrightarrow \quad \mathbf{I}_2 = -12\angle -60^\circ + 8 = 10.58\angle 79.1^\circ \text{ A}$$

For the voltage source, the current flowing from it is $\mathbf{I}_2 = 10.58\angle 79.1^\circ$ A and the voltage across it is $60\angle 30^\circ$ V, so that the average power is

$$P_5 = \frac{1}{2}(60)(10.58)\cos(30^\circ - 79.1^\circ) = 207.8 \text{ W}$$

Following the passive sign convention (see Fig. 1.8), this average power is absorbed by the source, in view of the direction of $\mathbf{I}_2$ and the polarity of the voltage source. That is, the circuit is delivering average power to the voltage source.

For the current source, the current through it is $\mathbf{I}_1 = 4\angle 0^\circ$ and the voltage across it is

$$\mathbf{V}_1 = 20\mathbf{I}_1 + j10(\mathbf{I}_1 - \mathbf{I}_2) = 80 + j10(4 - 2 - j10.39) = 183.9 + j20 = 184.984\angle 6.21^\circ \text{ V}$$

The average power supplied by the current source is

$$P_1 = -\frac{1}{2}(184.984)(4)\cos(6.21^\circ - 0) = -367.8 \text{ W}$$

It is negative according to the passive sign convention, meaning that the current source is supplying power to the circuit.

For the resistor, the current through it is $\mathbf{I}_1 = 4\angle 0^\circ$ and the voltage across it is $20\mathbf{I}_1 = 80\angle 0^\circ$, so that the power absorbed by the resistor is

$$P_2 = \frac{1}{2}(80)(4) = 160 \text{ W}$$

For the capacitor, the current through it is $\mathbf{I}_2 = 10.58\underline{/79.1^\circ}$ and the voltage across it is $-j5\mathbf{I}_2 = (5\underline{/-90^\circ})(10.58\underline{/79.1^\circ}) = 52.9\underline{/79.1^\circ - 90^\circ}$. The average power absorbed by the capacitor is

$$P_4 = \frac{1}{2}(52.9)(10.58)\cos(-90^\circ) = 0$$

For the inductor, the current through it is $\mathbf{I}_1 - \mathbf{I}_2 = 2 - j10.39 = 10.58\underline{/-79.1^\circ}$. The voltage across it is $j10(\mathbf{I}_1 - \mathbf{I}_2) = 105.8\underline{/-79.1^\circ + 90^\circ}$. Hence, the average power absorbed by the inductor is

$$P_3 = \frac{1}{2}(105.8)(10.58)\cos 90^\circ = 0$$

Notice that the inductor and the capacitor absorb zero average power and that the total power supplied by the current source equals the power absorbed by the resistor and the voltage source, or

$$P_1 + P_2 + P_3 + P_4 + P_5 = -367.8 + 160 + 0 + 0 + 207.8 = 0$$

indicating that power is conserved.

PRACTICE PROBLEM 11.4

Calculate the average power absorbed by each of the five elements in the circuit of Fig. 11.6.

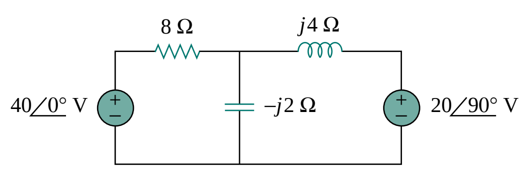

Figure 11.6 For Practice Prob. 11.4.

Answer: 40-V Voltage source: −100 W; resistor: 100 W; others: 0 W.

11.3 MAXIMUM AVERAGE POWER TRANSFER

In Section 4.8 we solved the problem of maximizing the power delivered by a power-supplying resistive network to a load R_L. Representing the circuit by its Thevenin equivalent, we proved that the maximum power would be delivered to the load if the load resistance is equal to the Thevenin resistance $R_L = R_{\text{Th}}$. We now extend that result to ac circuits.

Consider the circuit in Fig. 11.7, where an ac circuit is connected to a load $\mathbf{Z}_L$ and is represented by its Thevenin equivalent. The load is usually represented by an impedance, which may model an electric motor, an antenna, a TV, and so forth. In rectangular form, the Thevenin impedance $\mathbf{Z}_{\text{Th}}$ and the load impedance $\mathbf{Z}_L$ are

$$\mathbf{Z}_{\text{Th}} = R_{\text{Th}} + jX_{\text{Th}} \tag{11.13a}$$

$$\mathbf{Z}_L = R_L + jX_L \tag{11.13b}$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{Z}_{\text{Th}} + \mathbf{Z}_L} = \frac{\mathbf{V}_{\text{Th}}}{(R_{\text{Th}} + jX_{\text{Th}}) + (R_L + jX_L)} \tag{11.14}$$

From Eq. (11.11), the average power delivered to the load is

$$P = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{|\mathbf{V}_{\text{Th}}|^2 R_L/2}{(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2} \tag{11.15}$$

Our objective is to adjust the load parameters R_L and X_L so that P is maximum. To do this we set $\partial P/\partial R_L$ and $\partial P/\partial X_L$ equal to zero. From Eq. (11.15), we obtain

$$\frac{\partial P}{\partial X_L} = -\frac{|\mathbf{V}_{\text{Th}}|^2 R_L (X_{\text{Th}} + X_L)}{[(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2]^2} \tag{11.16a}$$

$$\frac{\partial P}{\partial R_L} = \frac{|\mathbf{V}_{\text{Th}}|^2 [(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2 - 2R_L(R_{\text{Th}} + R_L)]}{2[(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2]^2} \tag{11.16b}$$

Setting $\partial P/\partial X_L$ to zero gives

$$X_L = -X_{\text{Th}} \tag{11.17}$$

and setting $\partial P/\partial R_L$ to zero results in

$$R_L = \sqrt{R_{\text{Th}}^2 + (X_{\text{Th}} + X_L)^2} \tag{11.18}$$

Combining Eqs. (11.17) and (11.18) leads to the conclusion that for maximum average power transfer, $\mathbf{Z}_L$ must be selected so that $X_L = -X_{\text{Th}}$ and $R_L = R_{\text{Th}}$, i.e.,

$$\boxed{\mathbf{Z}_L = R_L + jX_L = R_{\text{Th}} - jX_{\text{Th}} = \mathbf{Z}_{\text{Th}}^*} \tag{11.19}$$

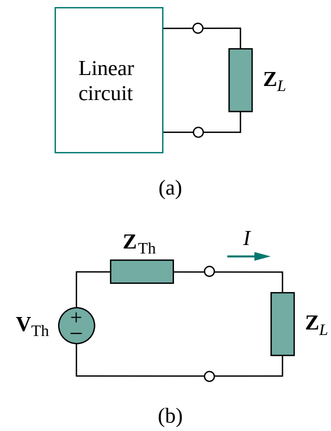

Figure 11.7 Finding the maximum average power transfer: (a) circuit with a load, (b) the Thevenin equivalent.

> For **maximum average power transfer**, the load impedance $\mathbf{Z}_L$ must be equal to the complex conjugate of the Thevenin impedance $\mathbf{Z}_{\text{Th}}$.

When $\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^*$, we say that the load is matched to the source.

This result is known as the *maximum average power transfer theorem* for the sinusoidal steady state. Setting $R_L = R_{\text{Th}}$ and $X_L = -X_{\text{Th}}$ in Eq. (11.15) gives us the maximum average power as

$$\boxed{P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{8R_{\text{Th}}}} \tag{11.20}$$

In a situation in which the load is purely real, the condition for maximum power transfer is obtained from Eq. (11.18) by setting $X_L = 0$; that is,

$$R_L = \sqrt{R_{\text{Th}}^2 + X_{\text{Th}}^2} = |\mathbf{Z}_{\text{Th}}| \tag{11.21}$$

This means that for maximum average power transfer to a purely resistive load, the load impedance (or resistance) is equal to the magnitude of the Thevenin impedance.

EXAMPLE 11.5

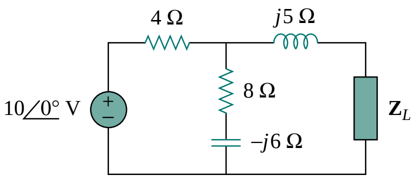

Figure 11.8 For Example 11.5.

Determine the load impedance $\mathbf{Z}_L$ that maximizes the average power drawn from the circuit of Fig. 11.8. What is the maximum average power?

Solution:

First we obtain the Thevenin equivalent at the load terminals. To get $\mathbf{Z}_{\text{Th}}$, consider the circuit shown in Fig. 11.9(a). We find

$$\mathbf{Z}_{\text{Th}} = j5 + 4 \parallel (8 - j6) = j5 + \frac{4(8 - j6)}{4 + 8 - j6} = 2.933 + j4.467\ \Omega$$

To find $\mathbf{V}_{\text{Th}}$, consider the circuit in Fig. 11.8(b). By voltage division,

$$\mathbf{V}_{\text{Th}} = \frac{8 - j6}{4 + 8 - j6}(10) = 7.454\underline{/-10.3^\circ}\text{ V}$$

The load impedance draws the maximum power from the circuit when

$$\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^* = 2.933 - j4.467\ \Omega$$

According to Eq. (11.20), the maximum average power is

$$P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{8R_{\text{Th}}} = \frac{(7.454)^2}{8(2.933)} = 2.368\text{ W}$$

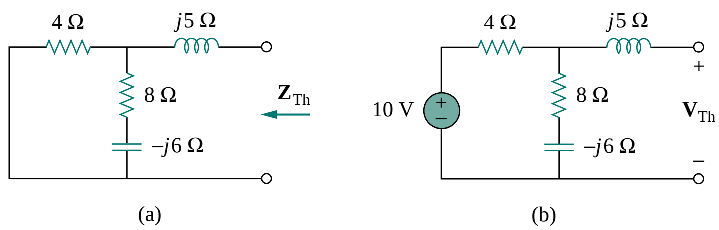

Figure 11.9 Finding the Thevenin equivalent of the circuit in Fig. 11.8.

PRACTICE PROBLEM 11.5

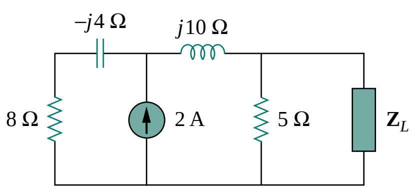

Figure 11.10 For Practice Prob. 11.5.

For the circuit shown in Fig. 11.10, find the load impedance $\mathbf{Z}_L$ that absorbs the maximum average power. Calculate that maximum average power.

Answer: $3.415 - j0.7317\ \Omega$, 1.429 W.

EXAMPLE 11.6

In the circuit in Fig. 11.11, find the value of R_L that will absorb the maximum average power. Calculate that power.

Solution:
We first find the Thevenin equivalent at the terminals of R_L.

$$\mathbf{Z}_{\text{Th}} = (40 - j30) \parallel j20 = \frac{j20(40 - j30)}{j20 + 40 - j30} = 9.412 + j22.35\ \Omega$$

By voltage division,

$$\mathbf{V}_{\text{Th}} = \frac{j20}{j20 + 40 - j30}(150\underline{/30^\circ}) = 72.76\underline{/134^\circ}\ \text{V}$$

The value of R_L that will absorb the maximum average power is

$$R_L = |\mathbf{Z}_{\text{Th}}| = \sqrt{9.412^2 + 22.35^2} = 24.25\ \Omega$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{Z}_{\text{Th}} + R_L} = \frac{72.76\underline{/134^\circ}}{33.39 + j22.35} = 1.8\underline{/100.2^\circ}\ \text{A}$$

The maximum average power absorbed by R_L is

$$P_{\max} = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{1}{2}(1.8)^2(24.25) = 39.29\ \text{W}$$

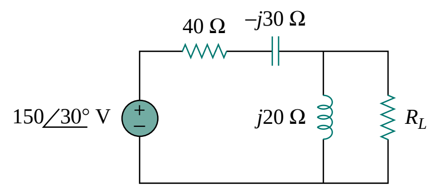

Figure 11.11 For Example 11.6.

PRACTICE PROBLEM 11.6

In Fig. 11.12, the resistor R_L is adjusted until it absorbs the maximum average power. Calculate R_L and the maximum average power absorbed by it.

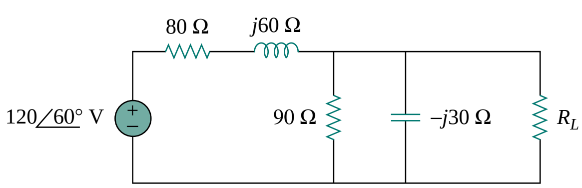

Figure 11.12 For Practice Prob. 11.6.

Answer: 30 Ω, 9.883 W.

11.4 EFFECTIVE OR RMS VALUE

The idea of *effective value* arises from the need to measure the effectiveness of a voltage or current source in delivering power to a resistive load.

The effective value of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current.

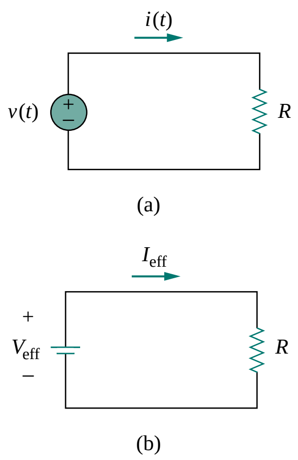

Figure 11.13 Finding the effective current: (a) ac circuit, (b) dc circuit.

In Fig. 11.13, the circuit in (a) is ac while that of (b) is dc. Our objective is to find I_{eff} that will transfer the same power to resistor R as the sinusoid i. The average power absorbed by the resistor in the ac circuit is

$$P = \frac{1}{T}\int_0^T i^2 R\, dt = \frac{R}{T}\int_0^T i^2\, dt \tag{11.22}$$

while the power absorbed by the resistor in the dc circuit is

$$P = I_{\text{eff}}^2 R \tag{11.23}$$

Equating the expressions in Eqs. (11.22) and (11.23) and solving for I_{eff}, we obtain

$$I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2\, dt} \tag{11.24}$$

The effective value of the voltage is found in the same way as current; that is,

$$V_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T v^2\, dt} \tag{11.25}$$

This indicates that the effective value is the (square) *root* of the *mean* (or average) of the *square* of the periodic signal. Thus, the effective value is often known as the *root-mean-square* value, or *rms* value for short; and we write

$$I_{\text{eff}} = I_{\text{rms}}, \qquad V_{\text{eff}} = V_{\text{rms}} \tag{11.26}$$

For any periodic function $x(t)$ in general, the rms value is given by

$$X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2\, dt} \tag{11.27}$$

The effective value of a periodic signal is its root mean square (rms) value.

Equation 11.27 states that to find the rms value of $x(t)$, we first find its *square* x^2 and then find the *mean* of that, or

$$\frac{1}{T}\int_0^T x^2\, dt$$

and then the square *root* ($\sqrt{\quad}$) of that mean. The rms value of a constant is the constant itself. For the sinusoid $i(t) = I_m \cos \omega t$, the effective or rms value is

$$\begin{aligned} I_{\text{rms}} &= \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2 \omega t\, dt} \\ &= \sqrt{\frac{I_m^2}{T}\int_0^T \frac{1}{2}(1 + \cos 2\omega t)\, dt} = \frac{I_m}{\sqrt{2}} \end{aligned} \tag{11.28}$$

Similarly, for $v(t) = V_m \cos \omega t$,

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}} \tag{11.29}$$

Keep in mind that Eqs. (11.28) and (11.29) are only valid for sinusoidal signals.

The average power in Eq. (11.8) can be written in terms of the rms values.

$$\begin{aligned} P &= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i) \\ &= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) \end{aligned} \tag{11.30}$$

Similarly, the average power absorbed by a resistor R in Eq. (11.11) can be written as

$$P = I_{\text{rms}}^2 R = \frac{V_{\text{rms}}^2}{R} \tag{11.31}$$

When a sinusoidal voltage or current is specified, it is often in terms of its maximum (or peak) value or its rms value, since its average value is zero. The power industries specify phasor magnitudes in terms of their rms values rather than peak values. For instance, the 110 V available at every household is the rms value of the voltage from the power company. It is convenient in power analysis to express voltage and current in their rms values. Also, analog voltmeters and ammeters are designed to read directly the rms value of voltage and current, respectively.

EXAMPLE 11.7

Determine the rms value of the current waveform in Fig. 11.14. If the current is passed through a 2-Ω resistor, find the average power absorbed by the resistor.

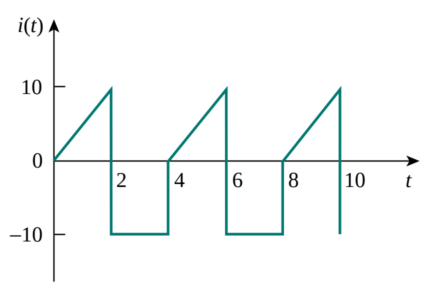

Figure 11.14 For Example 11.7.

Solution:

The period of the waveform is $T = 4$. Over a period, we can write the current waveform as

$$i(t) = \begin{cases} 5t, & 0 < t < 2 \\ -10, & 2 < t < 4 \end{cases}$$

The rms value is

$$\begin{aligned} I_{\text{rms}} &= \sqrt{\frac{1}{T}\int_0^T i^2\,dt} = \sqrt{\frac{1}{4}\left[\int_0^2 (5t)^2\,dt + \int_2^4 (-10)^2\,dt\right]} \\ &= \sqrt{\frac{1}{4}\left[25\frac{t^3}{3}\bigg|_0^2 + 100t\bigg|_2^4\right]} = \sqrt{\frac{1}{4}\left(\frac{200}{3} + 200\right)} = 8.165\text{ A} \end{aligned}$$

The power absorbed by a 2-Ω resistor is

$$P = I_{\text{rms}}^2 R = (8.165)^2(2) = 133.3\text{ W}$$

PRACTICE PROBLEM 11.7

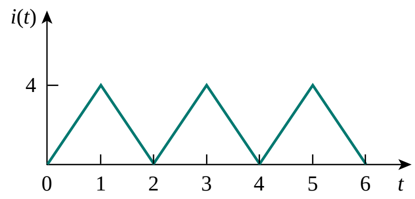

Figure 11.15 For Practice Prob. 11.7.

Find the rms value of the current waveform of Fig. 11.15. If the current flows through a 9-Ω resistor, calculate the average power absorbed by the resistor.

Answer: 2.309 A, 48 W.

EXAMPLE 11.8

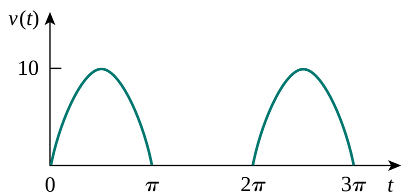

Figure 11.16 For Example 11.8.

The waveform shown in Fig. 11.16 is a half-wave rectified sine wave. Find the rms value and the amount of average power dissipated in a 10-Ω resistor.

Solution:

The period of the voltage waveform is $T = 2\pi$, and

$$v(t) = \begin{cases} 10\sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

The rms value is obtained as

$$V_{\text{rms}}^2 = \frac{1}{T}\int_0^T v^2(t)\,dt = \frac{1}{2\pi}\left[\int_0^{\pi} (10\sin t)^2\,dt + \int_{\pi}^{2\pi} 0^2\,dt\right]$$

But $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$. Hence

$$V_{\text{rms}}^2 = \frac{1}{2\pi}\int_0^{\pi} \frac{100}{2}(1 - \cos 2t)\,dt = \frac{50}{2\pi}\left(t - \frac{\sin 2t}{2}\right)\Bigg|_0^{\pi}$$

$$= \frac{50}{2\pi}\left(\pi - \frac{1}{2}\sin 2\pi - 0\right) = 25, \qquad V_{\text{rms}} = 5\text{ V}$$

The average power absorbed is

$$P = \frac{V_{\text{rms}}^2}{R} = \frac{5^2}{10} = 2.5\text{ W}$$

PRACTICE PROBLEM 11.8

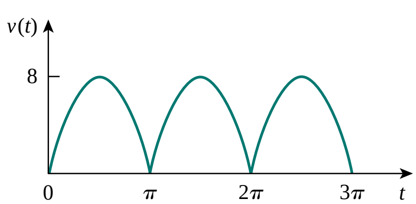

Figure 11.17 For Practice Prob. 11.8.

Find the rms value of the full-wave rectified sine wave in Fig. 11.17. Calculate the average power dissipated in a 6-Ω resistor.

Answer: 5.657 V, 5.334 W.

11.5 APPARENT POWER AND POWER FACTOR

In Section 11.2 we see that if the voltage and current at the terminals of a circuit are

$$v(t) = V_m \cos(\omega t + \theta_v) \qquad \text{and} \qquad i(t) = I_m \cos(\omega t + \theta_i) \tag{11.32}$$

or, in phasor form, $\mathbf{V} = V_m \angle \theta_v$ and $\mathbf{I} = I_m \angle \theta_i$, the average power is

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \tag{11.33}$$

In Section 11.4, we saw that

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) = S \cos(\theta_v - \theta_i) \tag{11.34}$$

We have added a new term to the equation:

$$\boxed{S = V_{\text{rms}} I_{\text{rms}}} \tag{11.35}$$

The average power is a product of two terms. The product $V_{\text{rms}} I_{\text{rms}}$ is known as the *apparent power* S. The factor $\cos(\theta_v - \theta_i)$ is called the *power factor* (pf).

> The **apparent power** (in VA) is the product of the rms values of voltage and current.

The apparent power is so called because it seems apparent that the power should be the voltage-current product, by analogy with dc resistive circuits. It is measured in volt-amperes or VA to distinguish it from the average or real power, which is measured in watts. The power factor is dimensionless, since it is the ratio of the average power to the apparent power,

$$\boxed{\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)} \tag{11.36}$$

The angle $\theta_v - \theta_i$ is called the *power factor angle*, since it is the angle whose cosine is the power factor. The power factor angle is equal to the angle of the load impedance if $\mathbf{V}$ is the voltage across the load and $\mathbf{I}$ is the current through it. This is evident from the fact that

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{V_m}{I_m} \angle \theta_v - \theta_i \tag{11.37}$$

Alternatively, since

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}} \angle \theta_v \tag{11.38a}$$

and

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}} \angle \theta_i \tag{11.38b}$$

the impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \angle \theta_v - \theta_i \tag{11.39}$$

> The power factor is the cosine of the phase difference between voltage and current. It is also the cosine of the angle of the load impedance.

From Eq. (11.36), the power factor may also be regarded as the ratio of the real power dissipated in the load to the apparent power of the load.

From Eq. (11.36), the power factor may be seen as that factor by which the apparent power must be multiplied to obtain the real or average power. The value of pf ranges between zero and unity. For a purely resistive load, the voltage and current are in phase, so that $\theta_v - \theta_i = 0$ and pf $= 1$. This implies that the apparent power is equal to the average power. For a purely reactive load, $\theta_v - \theta_i = \pm 90^\circ$ and pf $= 0$. In this case the average power is zero. In between these two extreme cases, pf is said to be *leading* or *lagging*. Leading power factor means that current leads voltage, which implies a capacitive load. Lagging power factor means that current lags voltage, implying an inductive load. Power factor affects the electric bills consumers pay the electric utility companies, as we will see in Section 11.9.2.

EXAMPLE 11.9

A series-connected load draws a current $i(t) = 4\cos(100\pi t + 10^\circ)$ A when the applied voltage is $v(t) = 120\cos(100\pi t - 20^\circ)$ V. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

Solution:

The apparent power is

$$S = V_{\text{rms}} I_{\text{rms}} = \frac{120}{\sqrt{2}} \frac{4}{\sqrt{2}} = 240 \text{ VA}$$

The power factor is

$$\text{pf} = \cos(\theta_v - \theta_i) = \cos(-20^\circ - 10^\circ) = 0.866 \qquad \text{(leading)}$$

The pf is leading because the current leads the voltage. The pf may also be obtained from the load impedance.

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{120\angle -20^\circ}{4\angle 10^\circ} = 30\angle -30^\circ = 25.98 - j15\ \Omega$$

$$\text{pf} = \cos(-30^\circ) = 0.866 \qquad \text{(leading)}$$

The load impedance $\mathbf{Z}$ can be modeled by a 25.98-Ω resistor in series with a capacitor with

$$X_C = -15 = -\frac{1}{\omega C}$$

or

$$C = \frac{1}{15\omega} = \frac{1}{15 \times 100\pi} = 212.2\ \mu\text{F}$$

PRACTICE PROBLEM 11.9

Obtain the power factor and the apparent power of a load whose impedance is $\mathbf{Z} = 60 + j40\ \Omega$ when the applied voltage is $v(t) = 150\cos(377t + 10^\circ)$ V.

Answer: 0.832 lagging, 156 VA.

EXAMPLE 11.10

Determine the power factor of the entire circuit of Fig. 11.18 as seen by the source. Calculate the average power delivered by the source.

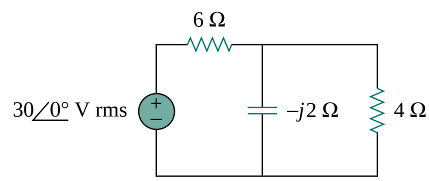

Figure 11.18 For Example 11.10.

Solution:

The total impedance is

$$\mathbf{Z} = 6 + 4 \parallel (-j2) = 6 + \frac{-j2 \times 4}{4 - j2} = 6.8 - j1.6 = 7\underline{/-13.24}\ \Omega$$

The power factor is

$$\text{pf} = \cos(-13.24) = 0.9734 \qquad \text{(leading)}$$

since the impedance is capacitive. The rms value of the current is

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{Z}} = \frac{30\underline{/0^\circ}}{7\underline{/-13.24^\circ}} = 4.286\underline{/13.24^\circ}\ \text{A}$$

The average power supplied by the source is

$$P = V_{\text{rms}} I_{\text{rms}}\ \text{pf} = (30)(4.286)0.9734 = 125\ \text{W}$$

or

$$P = I_{\text{rms}}^2 R = (4.286)^2(6.8) = 125\ \text{W}$$

where R is the resistive part of $\mathbf{Z}$.

PRACTICE PROBLEM 11.10

Calculate the power factor of the entire circuit of Fig. 11.19 as seen by the source. What is the average power supplied by the source?

Answer: 0.936 lagging, 118 W.

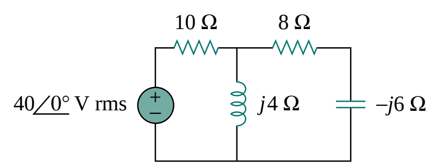

Figure 11.19 For Practice Prob. 11.10.

11.6 COMPLEX POWER

Considerable effort has been expended over the years to express power relations as simply as possible. Power engineers have coined the term *complex power*, which they use to find the total effect of parallel loads. Complex power is important in power analysis because it contains *all* the information pertaining to the power absorbed by a given load.

Consider the ac load in Fig. 11.20. Given the phasor form $\mathbf{V} = V_m\underline{/\theta_v}$ and $\mathbf{I} = I_m\underline{/\theta_i}$ of voltage $v(t)$ and current $i(t)$, the *complex power* $\mathbf{S}$ absorbed by the ac load is the product of the voltage and the complex conjugate of the current, or

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* \tag{11.40}$$

assuming the passive sign convention (see Fig. 11.20). In terms of the rms values,

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^* \tag{11.41}$$

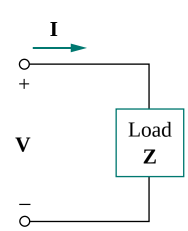

Figure 11.20 The voltage and current phasors associated with a load.

where

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}} \underline{/\theta_v} \tag{11.42}$$

and

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}} \underline{/\theta_i} \tag{11.43}$$

When working with the rms values of currents or voltages, we may drop the subscript rms if no confusion will be caused by doing so.

Thus we may write Eq. (11.41) as

$$\begin{aligned} \mathbf{S} &= V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i} \\ &= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) + j V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i) \end{aligned} \tag{11.44}$$

This equation can also be obtained from Eq. (11.9). We notice from Eq. (11.44) that the magnitude of the complex power is the apparent power; hence, the complex power is measured in volt-amperes (VA). Also, we notice that the angle of the complex power is the power factor angle.

The complex power may be expressed in terms of the load impedance $\mathbf{Z}$. From Eq. (11.37), the load impedance $\mathbf{Z}$ may be written as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \underline{/\theta_v - \theta_i} \tag{11.45}$$

Thus, $\mathbf{V}_{\text{rms}} = \mathbf{Z}\mathbf{I}_{\text{rms}}$. Substituting this into Eq. (11.41) gives

$$\boxed{\mathbf{S} = I_{\text{rms}}^2 \mathbf{Z} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*}} \tag{11.46}$$

Since $\mathbf{Z} = R + jX$, Eq. (11.46) becomes

$$\mathbf{S} = I_{\text{rms}}^2 (R + jX) = P + jQ \tag{11.47}$$

where P and Q are the real and imaginary parts of the complex power; that is,

$$P = \text{Re}(\mathbf{S}) = I_{\text{rms}}^2 R \tag{11.48}$$

$$Q = \text{Im}(\mathbf{S}) = I_{\text{rms}}^2 X \tag{11.49}$$

P is the average or real power and it depends on the load's resistance R. Q depends on the load's reactance X and is called the *reactive* (or quadrature) power.

Comparing Eq. (11.44) with Eq. (11.47), we notice that

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i), \qquad Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i) \tag{11.50}$$

The real power P is the average power in watts delivered to a load; it is the only useful power. It is the actual power dissipated by the load. The reactive power Q is a measure of the energy exchange between the source and the reactive part of the load. The unit of Q is the *volt-ampere reactive* (VAR) to distinguish it from the real power, whose unit is the watt. We know from Chapter 6 that energy storage elements neither dissipate nor supply power, but exchange power back and forth with the rest of the network. In the same way, the reactive power is being transferred back and forth between the load and the source. It represents a lossless interchange between the load and the source. Notice that:

1. $Q = 0$ for resistive loads (unity pf).
2. $Q < 0$ for capacitive loads (leading pf).
3. $Q > 0$ for inductive loads (lagging pf).

Thus,

> **Complex power** (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. As a complex quantity, its real part is real power P and its imaginary part is reactive power Q.

Introducing the complex power enables us to obtain the real and reactive powers directly from voltage and current phasors.

$$\begin{aligned} \text{Complex Power} &= \mathbf{S} = P + jQ = \frac{1}{2}\mathbf{V}\mathbf{I}^* \\ &= V_{\text{rms}} I_{\text{rms}} \angle \theta_v - \theta_i \\ \text{Apparent Power} &= S = |\mathbf{S}| = V_{\text{rms}} I_{\text{rms}} = \sqrt{P^2 + Q^2} \\ \text{Real Power} &= P = \text{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i) \\ \text{Reactive Power} &= Q = \text{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i) \\ \text{Power Factor} &= \frac{P}{S} = \cos(\theta_v - \theta_i) \end{aligned} \tag{11.51}$$

This shows how the complex power contains *all* the relevant power information in a given load.

S contains *all* power information of a load. The real part of **S** is the real power P; its imaginary part is the reactive power Q; its magnitude is the apparent power S; and the cosine of its phase angle is the power factor pf.

It is a standard practice to represent S, P, and Q in the form of a triangle, known as the *power triangle*, shown in Fig. 11.21(a). This is similar to the impedance triangle showing the relationship between $\mathbf{Z}$, R, and X, illustrated in Fig. 11.21(b). The power triangle has four items—the apparent/complex power, real power, reactive power, and the power factor angle. Given two of these items, the other two can easily be obtained from the triangle. As shown in Fig. 11.22, when $\mathbf{S}$ lies in the first quadrant, we have an inductive load and a lagging pf. When $\mathbf{S}$ lies in the fourth quadrant, the load is capacitive and the pf is leading. It is also possible for the complex power to lie in the second or third quadrant. This requires that the load impedance have a negative resistance, which is possible with active circuits.

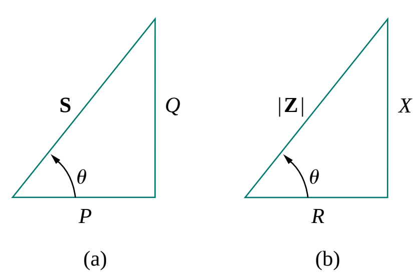

Figure 11.21 (a) Power triangle, (b) impedance triangle.

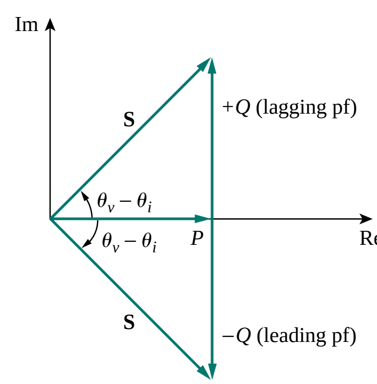

Figure 11.22 Power triangle.

EXAMPLE 11.11

The voltage across a load is $v(t) = 60\cos(\omega t - 10°)$ V and the current through the element in the direction of the voltage drop is $i(t) = 1.5\cos(\omega t + 50°)$ A. Find: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Solution:

(a) For the rms values of the voltage and current, we write

$$\mathbf{V}_{\text{rms}} = \frac{60}{\sqrt{2}}\angle -10°, \qquad \mathbf{I}_{\text{rms}} = \frac{1.5}{\sqrt{2}}\angle +50°$$

The complex power is

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^* = \left(\frac{60}{\sqrt{2}}\angle -10°\right)\left(\frac{1.5}{\sqrt{2}}\angle -50°\right) = 45\angle -60° \text{ VA}$$

The apparent power is

$$S = |\mathbf{S}| = 45 \text{ VA}$$

(b) We can express the complex power in rectangular form as

$$\mathbf{S} = 45\angle -60° = 45[\cos(-60°) + j\sin(-60°)] = 22.5 - j38.97$$

Since $\mathbf{S} = P + jQ$, the real power is

$$P = 22.5 \text{ W}$$

while the reactive power is

$$Q = -38.97 \text{ VAR}$$

(c) The power factor is

$$\text{pf} = \cos(-60°) = 0.5 \text{ (leading)}$$

It is leading, because the reactive power is negative. The load impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{60\angle -10°}{1.5\angle +50°} = 40\angle -60° \ \Omega$$

which is a capacitive impedance.

PRACTICE PROBLEM 11.11

For a load, $\mathbf{V}_{\text{rms}} = 110\angle 85°$ V, $\mathbf{I}_{\text{rms}} = 0.4\angle 15°$ A. Determine: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Answer: (a) $44\angle 70°$ VA, 44 VA, (b) 15.05 W, 41.35 VAR, (c) 0.342 lagging, $94.06 + j258.4\ \Omega$.

EXAMPLE 11.12

A load $\mathbf{Z}$ draws 12 kVA at a power factor of 0.856 lagging from a 120-V rms sinusoidal source. Calculate: (a) the average and reactive powers delivered to the load, (b) the peak current, and (c) the load impedance.

Solution:

(a) Given that pf $= \cos\theta = 0.856$, we obtain the power angle as $\theta = \cos^{-1} 0.856 = 31.13^\circ$. If the apparent power is $S = 12{,}000$ VA, then the average or real power is

$$P = S\cos\theta = 12{,}000 \times 0.856 = 10.272 \text{ kW}$$

while the reactive power is

$$Q = S\sin\theta = 12{,}000 \times 0.517 = 6.204 \text{ kVA}$$

(b) Since the pf is lagging, the complex power is

$$\mathbf{S} = P + jQ = 10.272 + j6.204 \text{ kVA}$$

From $\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}}$, we obtain

$$\mathbf{I}^*_{\text{rms}} = \frac{\mathbf{S}}{\mathbf{V}_{\text{rms}}} = \frac{10{,}272 + j6204}{120\underline{/0^\circ}} = 85.6 + j51.7 \text{ A} = 100\underline{/31.13^\circ} \text{ A}$$

Thus $\mathbf{I}_{\text{rms}} = 100\underline{/-31.13^\circ}$ and the peak current is

$$I_m = \sqrt{2}I_{\text{rms}} = \sqrt{2}(100) = 141.4 \text{ A}$$

(c) The load impedance

$$\mathbf{Z} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{120\underline{/0^\circ}}{100\underline{/-31.13^\circ}} = 1.2\underline{/31.13^\circ}\ \Omega$$

which is an inductive impedance.

PRACTICE PROBLEM 11.12

A sinusoidal source supplies 10 kVA reactive power to load $\mathbf{Z} = 250\underline{/-75^\circ}\ \Omega$. Determine: (a) the power factor, (b) the apparent power delivered to the load, and (c) the peak voltage.

Answer: (a) 0.2588 leading, (b) −10.35 kVAR, (c) 2.275 kV.

†11.7 CONSERVATION OF AC POWER

The principle of conservation of power applies to ac circuits as well as to dc circuits (see Section 1.5).

In fact, we already saw in Examples 11.3 and 11.4 that average power is conserved in ac circuits.

To see this, consider the circuit in Fig. 11.23(a), where two load impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are connected in parallel across an ac source $\mathbf{V}$. KCL gives

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 \tag{11.52}$$

The complex power supplied by the source is

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2}\mathbf{V}(\mathbf{I}_1^* + \mathbf{I}_2^*) = \frac{1}{2}\mathbf{V}\mathbf{I}_1^* + \frac{1}{2}\mathbf{V}\mathbf{I}_2^* = \mathbf{S}_1 + \mathbf{S}_2 \tag{11.53}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ denote the complex powers delivered to loads $\mathbf{Z}_1$ and $\mathbf{Z}_2$, respectively.

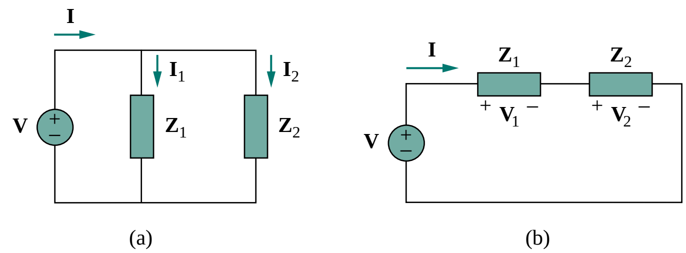

Figure 11.23 An ac voltage source supplied loads connected in: (a) parallel, (b) series.

If the loads are connected in series with the voltage source, as shown in Fig. 11.23(b), KVL yields

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 \tag{11.54}$$

The complex power supplied by the source is

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2}(\mathbf{V}_1 + \mathbf{V}_2)\mathbf{I}^* = \frac{1}{2}\mathbf{V}_1\mathbf{I}^* + \frac{1}{2}\mathbf{V}_2\mathbf{I}^* = \mathbf{S}_1 + \mathbf{S}_2 \tag{11.55}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ denote the complex powers delivered to loads $\mathbf{Z}_1$ and $\mathbf{Z}_2$, respectively.

We conclude from Eqs. (11.53) and (11.55) that whether the loads are connected in series or in parallel (or in general), the total power *supplied* by the source equals the total power *delivered* to the load. Thus, in general, for a source connected to N loads,

$$\boxed{\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_1 + \cdots + \mathbf{S}_N} \tag{11.56}$$

This means that the total complex power in a network is the sum of the complex powers of the individual components. (This is also true of real power and reactive power, but not true of apparent power.) This expresses the principle of conservation of ac power:

In fact, all forms of ac power are conserved: instantaneous, real, reactive, and complex.

> The complex, real, and reactive powers of the sources equal the respective sums of the complex, real, and reactive powers of the individual loads.

From this we imply that the real (or reactive) power flow from sources in a network equals the real (or reactive) power flow into the other elements in the network.

EXAMPLE 11.13

Figure 11.24 shows a load being fed by a voltage source through a transmission line. The impedance of the line is represented by the $(4 + j2)\ \Omega$ impedance and a return path. Find the real power and reactive power absorbed by: (a) the source, (b) the line, and (c) the load.

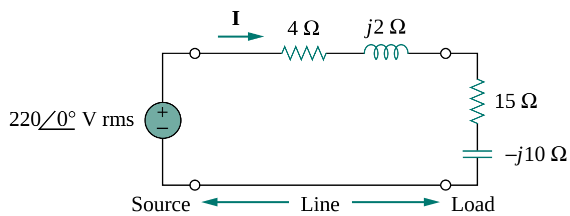

Figure 11.24 For Example 11.13.

Solution:

The total impedance is

$$\mathbf{Z} = (4 + j2) + (15 - j10) = 19 - j8 = 20.62\angle{-22.83^\circ}\ \Omega$$

The current through the circuit is

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{220\angle 0^\circ}{20.62\angle{-22.83^\circ}} = 10.67\angle 22.83^\circ \text{ A rms}$$

(a) For the source, the complex power is

$$\begin{aligned}\mathbf{S}_s = \mathbf{V}_s\mathbf{I}^* &= (220\angle 0^\circ)(10.67\angle{-22.83^\circ}) \\ &= 2347.4\angle{-22.83^\circ} = (2163.5 - j910.8) \text{ VA}\end{aligned}$$

From this, we obtain the real power as 2163.5 W and the reactive power as 910.8 VAR (leading).

(b) For the line, the voltage is

$$\begin{aligned}\mathbf{V}_{\text{line}} = (4 + j2)\mathbf{I} &= (4.472\angle 26.57^\circ)(10.67\angle 22.83^\circ) \\ &= 47.72\angle 49.4^\circ \text{ V rms}\end{aligned}$$

The complex power absorbed by the line is

$$\begin{aligned}\mathbf{S}_{\text{line}} = \mathbf{V}_{\text{line}}\mathbf{I}^* &= (47.72\angle 49.4^\circ)(10.67\angle{-22.83^\circ}) \\ &= 509.2\angle 26.57^\circ = 455.4 + j227.7 \text{ VA}\end{aligned}$$

or

$$\mathbf{S}_{\text{line}} = |\mathbf{I}|^2\mathbf{Z}_{\text{line}} = (10.67)^2(4 + j2) = 455.4 + j227.7 \text{ VA}$$

That is, the real power is 455.4 W and the reactive power is 227.76 VAR (lagging).

(c) For the load, the voltage is

$$\begin{aligned}\mathbf{V}_L = (15 - j10)\mathbf{I} &= (18.03\angle{-33.7^\circ})(10.67\angle 22.83^\circ) \\ &= 192.38\angle{-10.87^\circ} \text{ V rms}\end{aligned}$$

The complex power absorbed by the load is

$$\begin{aligned}\mathbf{S}_L = \mathbf{V}_L\mathbf{I}^* &= (192.38\angle{-10.87^\circ})(10.67\angle{-22.83^\circ}) \\ &= 2053\angle{-33.7^\circ} = (1708 - j1139) \text{ VA}\end{aligned}$$

We may cross check the result by finding the complex power $\mathbf{S}_s$ supplied by the source.

$$\mathbf{I}_t = \mathbf{I}_1 + \mathbf{I}_2 = (1.532 + j1.286) + (2.457 - j1.721)$$
$$= 4 - j0.435 = 4.024\underline{/-6.21^\circ} \text{ A rms}$$
$$\mathbf{S}_s = \mathbf{V}\mathbf{I}_t^* = (120\underline{/10^\circ})(4.024\underline{/6.21^\circ})$$
$$= 482.88\underline{/16.21^\circ} = 463 + j135 \text{ VA}$$

which is the same as before.

PRACTICE PROBLEM 11.14

Two loads connected in parallel are respectively 2 kW at a pf of 0.75 leading and 4 kW at a pf of 0.95 lagging. Calculate the pf of the two loads. Find the complex power supplied by the source.

Answer: 0.9972 (leading), $6 - j0.4495$ kVA.

11.8 POWER FACTOR CORRECTION

Most domestic loads (such as washing machines, air conditioners, and refrigerators) and industrial loads (such as induction motors) are inductive and operate at a low lagging power factor. Although the inductive nature of the load cannot be changed, we can increase its power factor.

> The process of increasing the power factor without altering the voltage or current to the original load is known as power factor correction.

Alternatively, power factor correction may be viewed as the addition of a reactive element (usually a capacitor) in parallel with the load in order to make the power factor closer to unity.

Since most loads are inductive, as shown in Fig. 11.27(a), a load's power factor is improved or corrected by deliberately installing a capacitor in parallel with the load, as shown in Fig. 11.27(b). The effect of adding the capacitor can be illustrated using either the power triangle or the phasor diagram of the currents involved. Figure 11.28 shows the latter, where it is assumed that the circuit in Fig. 11.27(a) has a power factor of $\cos\theta_1$, while the one in Fig. 11.27(b) has a power factor of $\cos\theta_2$. It is

An inductive load is modeled as a series combination of an inductor and a resistor.

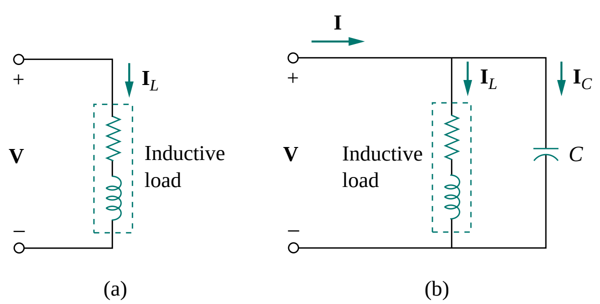

Figure 11.27 Power factor correction: (a) original inductive load, (b) inductive load with improved power factor.

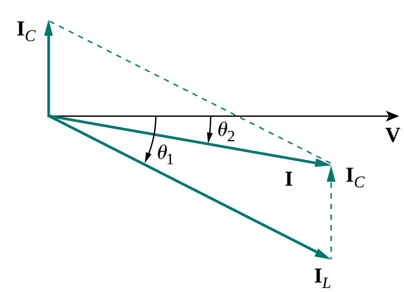

Figure 11.28 Phasor diagram showing the effect of adding a capacitor in parallel with the inductive load.

evident from Fig. 11.28 that adding the capacitor has caused the phase angle between the supplied voltage and current to reduce from θ_1 to θ_2, thereby increasing the power factor. We also notice from the magnitudes of the vectors in Fig. 11.28 that with the same supplied voltage, the circuit in Fig. 11.27(a) draws larger current I_L than the current I drawn by the circuit in Fig. 11.27(b). Power companies charge more for larger currents, because they result in increased power losses (by a squared factor, since $P = I_L^2 R$). Therefore, it is beneficial to both the power company and the consumer that every effort is made to minimize current level or keep the power factor as close to unity as possible. By choosing a suitable size for the capacitor, the current can be made to be completely in phase with the voltage, implying unity power factor.

We can look at the power factor correction from another perspective. Consider the power triangle in Fig. 11.29. If the original inductive load has apparent power S_1, then

$$P = S_1 \cos\theta_1, \qquad Q_1 = S_1 \sin\theta_1 = P \tan\theta_1 \tag{11.57}$$

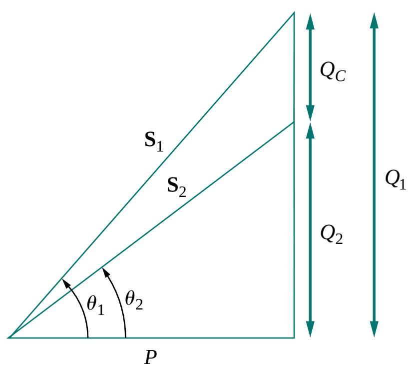

Figure 11.29 Power triangle illustrating power factor correction.

If we desire to increase the power factor from $\cos\theta_1$ to $\cos\theta_2$ without altering the real power (i.e., $P = S_2 \cos\theta_2$), then the new reactive power is

$$Q_2 = P \tan\theta_2 \tag{11.58}$$

The reduction in the reactive power is caused by the shunt capacitor, that is,

$$Q_C = Q_1 - Q_2 = P(\tan\theta_1 - \tan\theta_2) \tag{11.59}$$

But from Eq. (11.49), $Q_C = V_{\text{rms}}^2/X_C = \omega C V_{\text{rms}}^2$. The value of the required shunt capacitance C is determined as

$$C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{P(\tan\theta_1 - \tan\theta_2)}{\omega V_{\text{rms}}^2} \tag{11.60}$$

Note that the real power P dissipated by the load is not affected by the power factor correction because the average power due to the capacitance is zero.

Although the most common situation in practice is that of an inductive load, it is also possible that the load is capacitive, that is, the load is operating at a leading power factor. In this case, an inductor should be connected across the load for power factor correction. The required shunt inductance L can be calculated from

$$Q_L = \frac{V_{\text{rms}}^2}{X_L} = \frac{V_{\text{rms}}^2}{\omega L} \qquad \Longrightarrow \qquad L = \frac{V_{\text{rms}}^2}{\omega Q_L} \tag{11.61}$$

where $Q_L = Q_1 - Q_2$, the difference between the new and old reactive powers.

EXAMPLE 11.15

When connected to a 120-V (rms), 60-Hz power line, a load absorbs 4 kW at a lagging power factor of 0.8. Find the value of capacitance necessary to raise the pf to 0.95.

Solution:

If the pf = 0.8, then

$$\cos\theta_1 = 0.8 \qquad \Longrightarrow \qquad \theta_1 = 36.87^\circ$$

where θ_1 is the phase difference between voltage and current. We obtain the apparent power from the real power and the pf as

$$S_1 = \frac{P}{\cos\theta_1} = \frac{4000}{0.8} = 5000 \text{ VA}$$

The reactive power is

$$Q_1 = S_1 \sin\theta = 5000 \sin 36.87 = 3000 \text{ VAR}$$

When the pf is raised to 0.95,

$$\cos\theta_2 = 0.95 \qquad \Longrightarrow \qquad \theta_2 = 18.19^\circ$$

The real power P has not changed. But the apparent power has changed; its new value is

$$S_2 = \frac{P}{\cos\theta_2} = \frac{4000}{0.95} = 4210.5 \text{ VA}$$

The new reactive power is

$$Q_2 = S_2 \sin\theta_2 = 1314.4 \text{ VAR}$$

The difference between the new and old reactive powers is due to the parallel addition of the capacitor to the load. The reactive power due to the capacitor is

$$Q_C = Q_1 - Q_2 = 3000 - 1314.4 = 1685.6 \text{ VAR}$$

and

$$C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{1685.6}{2\pi \times 60 \times 120^2} = 310.5\ \mu\text{F}$$

PRACTICE PROBLEM 11.15

Find the value of parallel capacitance needed to correct a load of 140 kVAR at 0.85 lagging pf to unity pf. Assume that the load is supplied by a 110-V (rms), 60-Hz line.

Answer: 30.69 mF.

†11.9 APPLICATIONS

In this section, we consider two important application areas: how power is measured and how electric utility companies determine the cost of electricity consumption.

Reactive power is measured by an instrument called the *varmeter*. The varmeter is often connected to the load in the same way as the wattmeter.

11.9.1 Power Measurement

The average power absorbed by a load is measured by an instrument called the *wattmeter*.

The **wattmeter** is the instrument for measuring the average power.

Some wattmeters do not have coils; the wattmeter considered here is the electromagnetic type.

Figure 11.30 shows a wattmeter that consists essentially of two coils: the current coil and the voltage coil. A current coil with very low impedance (ideally zero) is connected in series with the load (Fig. 11.31) and responds to the load current. The voltage coil with very high impedance (ideally infinite) is connected in parallel with the load as shown in Fig. 11.31 and responds to the load voltage. The current coil acts like a short circuit because of its low impedance; the voltage coil behaves like an open circuit because of its high impedance. As a result, the presence of the wattmeter does not disturb the circuit or have an effect on the power measurement.

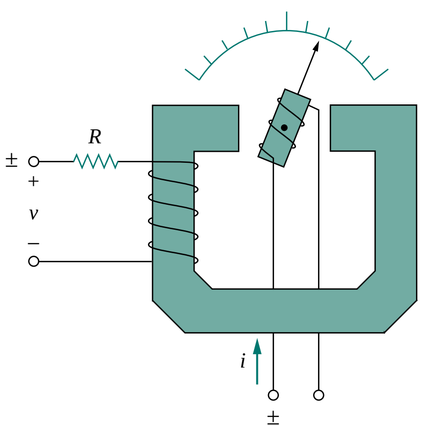

Figure 11.30 A wattmeter.

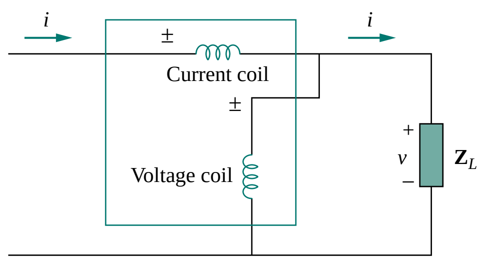

Figure 11.31 The wattmeter connected to the load.

When the two coils are energized, the mechanical inertia of the moving system produces a deflection angle that is proportional to the average value of the product $v(t)i(t)$. If the current and voltage of the load are $v(t) = V_m \cos(\omega t + \theta_v)$ and $i(t) = I_m \cos(\omega t + \theta_i)$, their corresponding rms phasors are

$$\mathbf{V}_{\text{rms}} = \frac{V_m}{\sqrt{2}} \angle \theta_v \qquad \text{and} \qquad \mathbf{I}_{\text{rms}} = \frac{I_m}{\sqrt{2}} \angle \theta_i \tag{11.62}$$

and the wattmeter measures the average power given by

$$P = |\mathbf{V}_{\text{rms}}||\mathbf{I}_{\text{rms}}| \cos(\theta_v - \theta_i) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \tag{11.63}$$

As shown in Fig. 11.31, each wattmeter coil has two terminals with one marked ±. To ensure upscale deflection, the ± terminal of the current coil is toward the source, while the ± terminal of the voltage coil is connected to the same line as the current coil. Reversing both coil connections still results in upscale deflection. However, reversing one coil and not the other results in downscale deflection and no wattmeter reading.

EXAMPLE 11.16

Find the wattmeter reading of the circuit in Fig. 11.32.

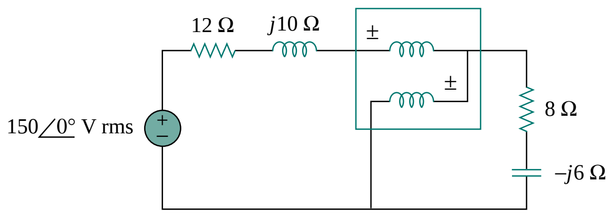

Figure 11.32 For Example 11.16.

Solution:

In Fig. 11.32, the wattmeter reads the average power absorbed by the $(8 - j6)$ Ω impedance because the current coil is in series with the impedance while the voltage coil is in parallel with it. The current through the circuit is

$$\mathbf{I} = \frac{150\angle 0^\circ}{(12 + j10) + (8 - j6)} = \frac{150}{20 + j4} \text{ A rms}$$

The voltage across the $(8 - j6)$ Ω impedance is

$$\mathbf{V} = \mathbf{I}(8 - j6) = \frac{150(8 - j6)}{20 + j4} \text{ V rms}$$

The complex power is

$$\mathbf{S} = \mathbf{VI}^* = \frac{150(8 - j6)}{20 + j4} \cdot \frac{150}{20 - j4} = \frac{150^2(8 - j6)}{20^2 + 4^2}$$
$$= 423.7 - j324.6 \text{ VA}$$

The wattmeter reads

$$P = \text{Re}(\mathbf{S}) = 432.7 \text{ W}$$

PRACTICE PROBLEM 11.16

For the circuit in Fig. 11.33, find the wattmeter reading.

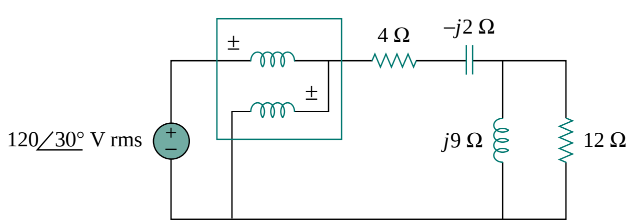

Figure 11.33 For Practice Prob. 11.16.

Answer: 1437 W.

11.9.2 Electricity Consumption Cost

In Section 1.7, we considered a simplified model of the way the cost of electricity consumption is determined. But the concept of power factor was not included in the calculations. Now we consider the importance of power factor in electricity consumption cost.

Loads with low power factors are costly to serve because they require large currents, as explained in Section 11.8. The ideal situation would be to draw minimum current from a supply so that $S = P$, $Q = 0$, and pf $= 1$. A load with nonzero Q means that energy flows forth and back between the load and the source, giving rise to additional power losses. In view of this, power companies often encourage their customers to have power factors as close to unity as possible and penalize some customers who do not improve their load power factors.

Utility companies divide their customers into categories: as residential (domestic), commercial, and industrial, or as small power, medium power, and large power. They have different rate structures for each category. The amount of energy consumed in units of kilowatt-hours (kWh) is measured using a kilowatt-hour meter installed at the customer's premises.

Although utility companies use different methods for charging customers, the tariff or charge to a consumer is often two-part. The first part is fixed and corresponds to the cost of generation, transmission, and distribution of electricity to meet the load requirements of the consumers. This part of the tariff is generally expressed as a certain price per kW of maximum demand. Or it may instead be based on kVA of maximum demand, to account for the power factor (pf) of the consumer. A pf penalty charge may be imposed on the consumer whereby a certain percentage of kW or kVA maximum demand is charged for every 0.01 fall in pf below a prescribed value, say 0.85 or 0.9. On the other hand, a pf credit may be given for every 0.01 that the pf exceeds the prescribed value.

The second part is proportional to the energy consumed in kWh; it may be in graded form, for example, the first 100 kWh at 16 cents/kWh, the next 200 kWh at 10 cents/kWh and so forth. Thus, the bill is determined based on the following equation:

$$\text{Total Cost} = \text{Fixed Cost} + \text{Cost of Energy} \tag{11.64}$$

EXAMPLE 11.17

A manufacturing industry consumes 200 MWh in one month. If the maximum demand is 1600 kW, calculate the electricity bill based on the following two-part rate:

Demand charge: \$5.00 per month per kW of billing demand.

Energy charge: 8 cents per kWh for the first 50,000 kWh, 5 cents per kWh for the remaining energy.

Solution:

The demand charge is

$$\$5.00 \times 1600 = \$8000 \tag{11.17.1}$$

The energy charge for the first 50,000 kWh is