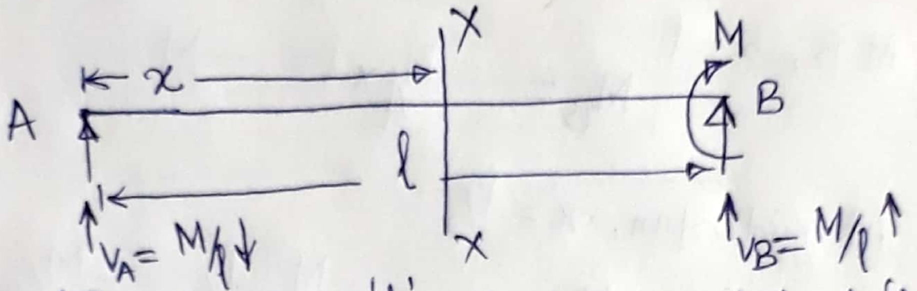
1. A Simply supported beam subjected to a moment at one support



A beam AB of span 'l' simply supported at 'A' & 'B' is subjected to moment 'M' (clockwise) at support 'B'. Let the vertical reactions at supports by V_A & V_B (both upwards).

$$\Sigma V = 0 \Rightarrow V_A + V_B = 0 \Rightarrow V_A = -V_B$$

The negative sign indicates that one of the reactions is downwards. Both V_A & V_B are of same magnitude.

$$\Sigma M = 0 \text{ @ } A \Rightarrow V_B \times l - M = 0 \Rightarrow V_B = M/l$$

Hence $V_A = -V_B = -M/l$

Thus reaction at A is M/l (downwards) while reaction at B is M/l (upwards)

Take section X–X at 'x' from A.

B.M. at X–X $\quad M_x = -M/l \cdot x$

Suggesting that the B.M. variation is linear and it has a maximum value for maximum value of x. Also, since it is negative, it is hogging throughout.

At A, $x=0$

$$M_A = -\frac{M}{l}\times 0 = 0$$

At B, $x=l$

$$M_B = -\frac{M}{l}\times l = -M$$

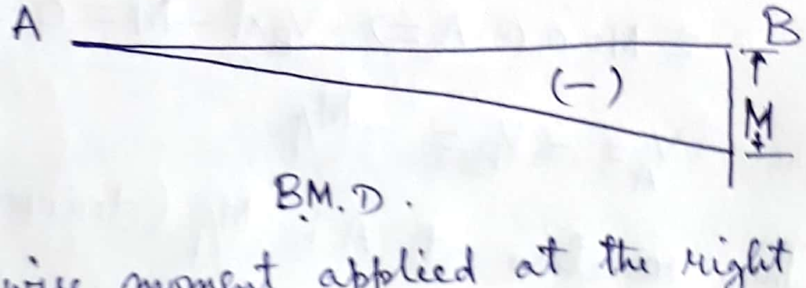
At mid span, $x = l/2$

$$M_{\text{mid span}} = -\frac{M}{l}\times l/2 = -M/2$$

Thus B.M. is a linear variation, maximum B.M. being the B.M. at B = $-M$ (the applied moment).

A clockwise moment 'M' applied at 'B' causes a B.M. equal to the moment itself at 'B' i.e. $M_B = -M$.

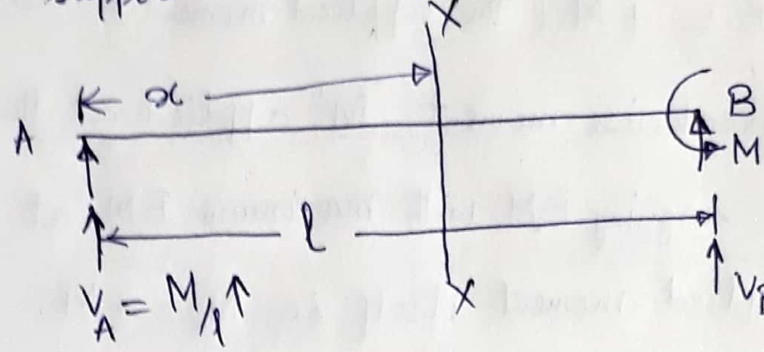
The B.M. diagram can be drawn as



B.M.D.

A clockwise moment applied at the right hand support causes a hogging B.M. throughout.

2. A Simply supported beam subjected to a moment at one support.



A beam AB of span 'l' simply supported at 'A' & 'B' is subjected to an anticlockwise moment 'M' at the right hand support 'B'. Let the vertical reactions at 'A' and 'B' be V_A & V_B (both upwards) as shown.

$\Sigma V = 0 \Rightarrow V_A = -V_B$ (Both of same magnitude but opposite in direction).

Also $\Sigma M = 0$ @ A $\Rightarrow V_B \times l + M = 0 \Rightarrow V_B = -M/l$

$V_A = -V_B = +M/l$

The reaction at 'A' is M/l (upwards) while reaction at 'B' is $-M/l$ i.e, M/l (downwards).

Take section X–X at 'x' from 'A'.

B.M. at X–X $\quad M_x = +\frac{M}{l} \cdot x$

The B.M. variation is linear with a maximum value for maximum value of x. Also it is sagging (+ve) throughout.

At $x = 0$ i.e, at A, $M_A = +M/l \times 0 = 0$

At $x = l$ i.e, at B, $M_B = +M/l \times l = +M$

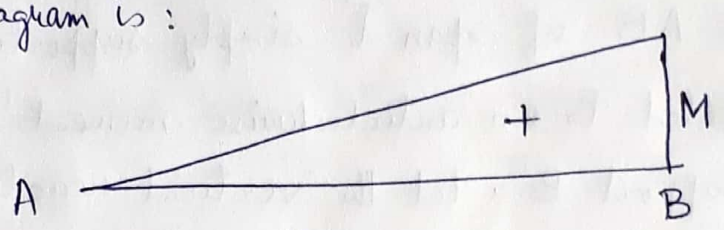
A $x = l/2$ i.e, at mid span,

$$M_{mid\ span} = +M/l \times l/2 = +M/2$$

The B.M. is, thus, a linear variation, maximum bending moment being at B = +M (the applied moment).

An anticlockwise moment 'M' applied at the right hand support causes a sagging B.M. with maximum B.M. at 'B' which is equal to the applied moment itself i.e. $M_B = +M$.

The B.M. diagram is:



B.M.D.

If the moment is applied at the left hand support 'A', it will cause opposite effect. Thus, we conclude as:

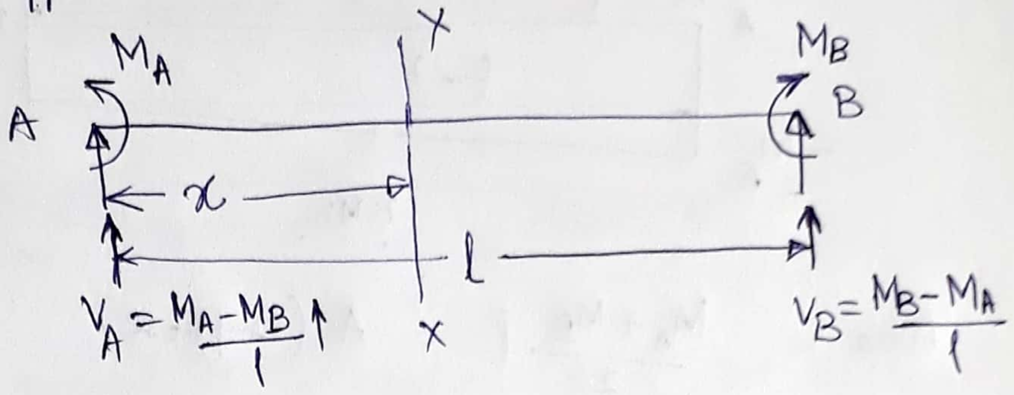
(i) A clockwise moment applied at the right hand support causes a linear hogging B.M. with BM at the right hand support = −M.

(ii) An anticlockwise moment applied at the right hand support causes a linear sagging B.M. with B.M. at the right hand support = +M

(iii) A clockwise moment applied at the left hand support causes a linear sagging B.M. with B.M. at left hand support = +M

(iv) An anticlockwise moment 'M' applied at the left hand support causes a linear hogging B.M. with B.M. at left hand support = −M.

3. A simply supported beam subjected to moments at both supports.



A simply supported beam 'AB' of span 'l' is subjected to moments M_A at A and M_B at B as shown.

The vertical reactions at A and B i.e, V_A & V_B are assumed to be upwards (both)

$$\Sigma V = 0 \Rightarrow V_A + V_B = 0 \Rightarrow V_A = -V_B$$

Also $\Sigma M = 0$ @ A $\Rightarrow V_B \times l + M_A - M_B = 0$

$$\Rightarrow V_B = \frac{M_B - M_A}{l} \uparrow$$

$$V_A = -V_B = -\left[\frac{M_B - M_A}{l}\right] = \frac{M_A - M_B}{l} \uparrow$$

Bending moment at X-X

$$M_x = \frac{M_A - M_B}{l} \cdot x - M_A$$

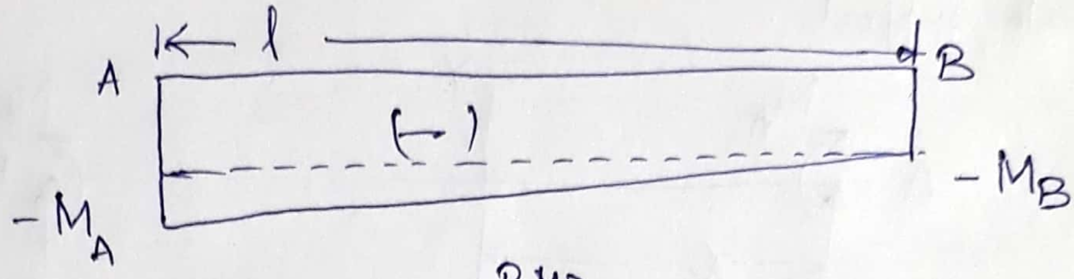
At A, $x = 0$ B.M. $= -M_A$

At B, $x = l$ B.M. $= \frac{M_A - M_B}{l} \cdot l - M_A = -M_B$

At mid span, $x = l/2$ B.M. $= \frac{M_A - M_B}{l} \cdot l/2 - M_A$

$$= -\frac{(M_A + M_B)}{2}$$

Bending moment diagram is



BMD.

$$\text{Area} = \frac{M_A + M_B}{2} \cdot l = A \text{ (suppose)}$$

If centroidal distance is $\bar{x}$ from A, then

$$A\bar{x} = M_B \cdot l \cdot l/2 + 1/2\,(M_A - M_B) \cdot l \cdot l/3$$

$$= \frac{l^2}{2} M_B + \frac{l^2}{6} M_A - \frac{l^2}{6} \cdot M_B$$

$$= l^2/6 \left[3M_B + M_A - M_B \right]$$

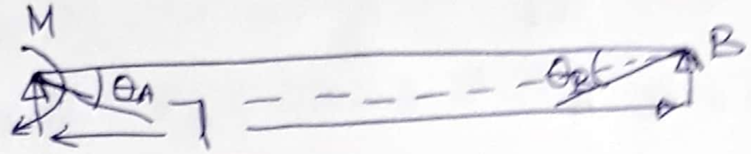
$$A\bar{x} = l^2/6 \left[M_A + 2M_B \right]$$

Similarly, it can be shown that $A\bar{x}'$ (where $\bar{x}'$ is the centroidal diance from B is

$$A\bar{x}' = l^2/6 \left[2M_A + M_B \right]$$

Stiffness: (k)

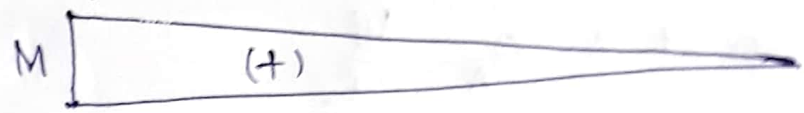
(i) A beam with far end free



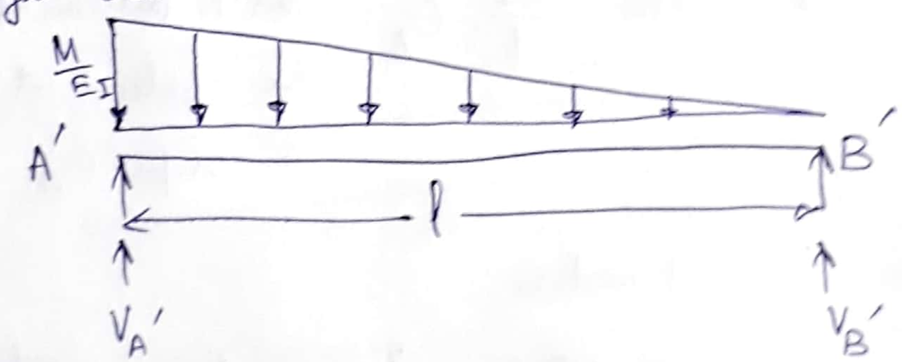
When a moment 'M' is applied at A, let the slope caused at 'A' be θ_A and that at B be θ_B.

Moment induced at $B = 0$ since 'B' is free to rotate.

The bending moment diagram for the beam AB is



conjugate beam is



As per conjugate beam principles,

$$\text{Slope at } A = \theta_A = \text{S.F. at } A'$$
$$= \text{Vertical reaction at } A'$$

And $$\text{Slope at } B = \theta_B = \text{S.F. at } B'$$
$$= \text{Vert. reaction at } B'$$

Using $\Sigma M = 0$ @ B

$$V_A' \times l - \frac{1}{2} \cdot \frac{M}{EI} \cdot l \cdot \frac{2l}{3} = 0$$

$$\Rightarrow V_A' = \frac{Ml}{3EI} = \theta_A$$

Also $\Sigma M = 0$ @ A

$$V_B' \times l - \frac{1}{2} \cdot \frac{M}{EI} \cdot l \cdot l/3 = 0$$

$$\Rightarrow V_B' = \frac{Ml}{6EI} = \theta_B = \frac{\theta_A}{2}$$

Thus Slope at A $\theta_A = \frac{Ml}{3EI}$ & slope at B $\theta_B = \frac{Ml}{6EI} = \frac{\theta_A}{2}$

Also Slope at A $\theta_A = \frac{Ml}{3EI}$

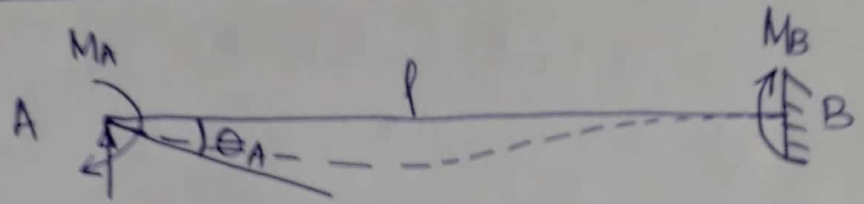
or $M = \frac{3EI}{l} \cdot \theta_A$ This is the moment required to be applied at A to cause a slope θ_A at A.

For $\theta_A = 1$ radian

$M = \frac{3EI}{l}$ This is the moment required to be applied at A to cause a unit slope at A.

Thus stiffness of beam AB at A $k_{AB} = \frac{3EI}{l}$ the far end 'B' being simply supported.

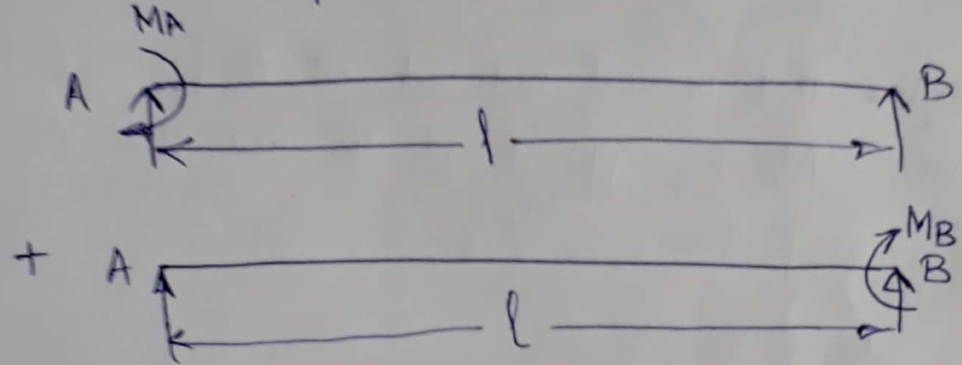
(ii) A beam with far end fixed.



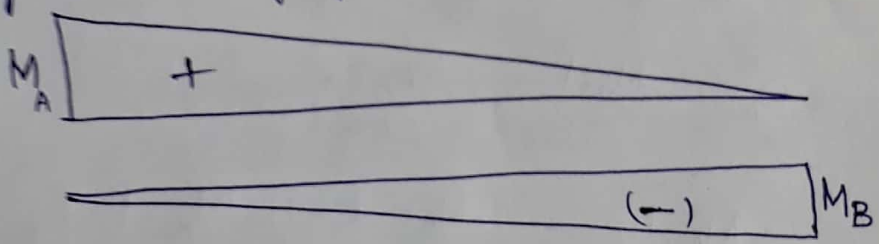
Let a moment M_A be applied at A. Since end B is fixed, a moment M_B is induced at B.

Slope at $B = 0$. Let slope caused at $A = \theta_A$.

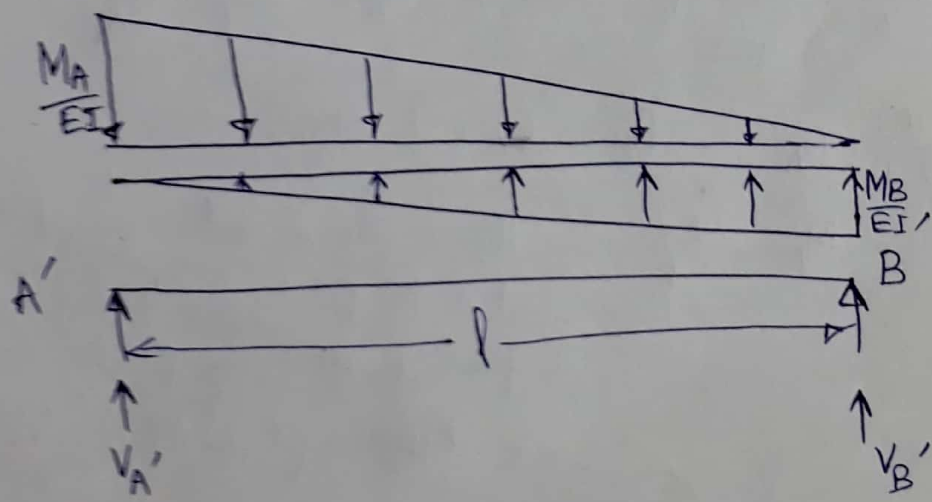
The beam can be represented as



The bending moment diagram/s can be drawn as



Using conjugate beam approach,



Slope at B $\theta_B = 0$ means S.F. at B = 0

or vert. reaction at B $V_{B'} = 0$

$\Sigma M = 0$ @ A'

$$V_{B'} \times l - \frac{1}{2} \cdot \frac{M_A}{EI} \cdot l \cdot \frac{l}{3} + \frac{1}{2} \frac{M_B}{EI} \cdot l \cdot \frac{2l}{3} = 0$$

$$\Rightarrow \quad V_{B'} = \frac{l}{6EI}\left[M_A - 2M_B\right] = 0 \quad [\because \theta_B = 0]$$

or $\quad M_B = M_A/2$

This means that moment induced at the far end 'B' when a moment M_A is applied at the near end 'A' is $M_A/2$.

carry over factor $= \frac{M_A}{2}/M_A = \frac{1}{2}$.

Also $\Sigma M = 0$ @ B'

$$V_{A'} \times l - \frac{1}{2} \frac{M_A}{EI} \cdot l \cdot \frac{2l}{3} + \frac{1}{2} \frac{M_B}{EI} \cdot l \cdot l/3 = 0$$

$$V_{A'} = \theta_A = \frac{M_A . l}{3EI} - \frac{M_B . l}{6EI}$$

$$= M_A . l/3EI - \frac{M_A/2 . l}{6EI} = \frac{M_A . l}{3EI}\left[1 - \frac{1}{4}\right] = \frac{M_A . l}{4EI}$$

$$\theta_A = \frac{M_A}{4EI} . l \quad \Rightarrow M_A = \frac{4EI}{l} . \theta_A$$

$M_A = \frac{4EI}{l} . \theta_A$ for $\theta_A = 1$ radian

$M_A = \frac{4EI}{l}$ – This is the moment required to be applied at 'A' to produce a unit slope at 'A', the far end 'B' being fixed

= Stiffness of beam AB = k_{AB}.

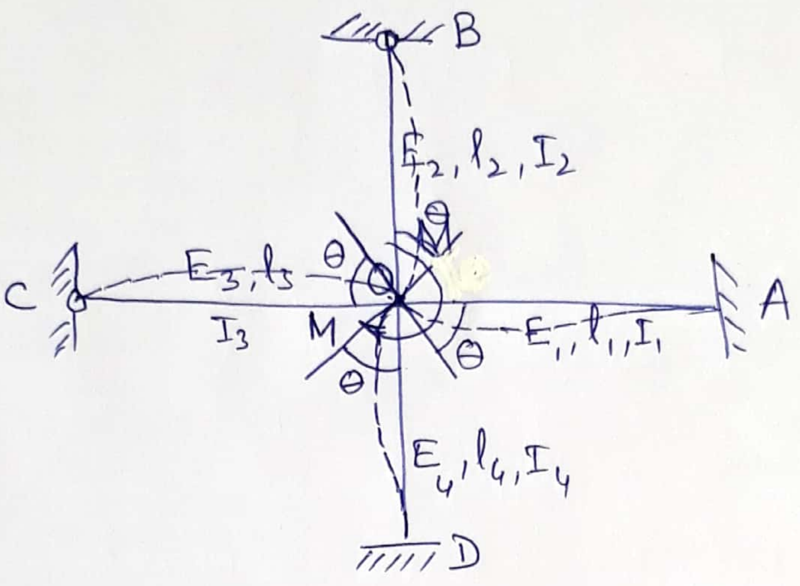


Figure 9 – Distribution Factors

Let there be a rigid joint 'O' where four members OA, OB, OC & CD are connected. Let ends 'A' and 'D' of members OA & OD be fixed and ends 'B' & 'C' of members OB & OC be hinged as shown in the figure. A clockwise moment 'M' is applied at the joint and the joint rotates through angle θ. Since it is a rigid joint, all the four members OA, OB, OC & OD meeting at the joint 'O' rotate through the same angle 'θ'. Let the moments shared by members OA, OB, OC & CD be m_1, m_2, m_3, m_4 respectively, then

$$M = m_1 + m_2 + m_3 + m_4 \quad \text{— (I)}$$

Also if k_1, k_2, k_3, k_4 are the stiffnesses of the members OA, OB, OC & OD respectively, then

$$m_1 = k_1 \cdot \theta$$

$$m_2 = k_2 \cdot \theta$$

$$m_3 = k_3 \cdot \theta$$

$$m_4 = k_4 \cdot \theta$$

Substituting these values in the equation I

$$k_1\theta + k_2\theta + k_3\theta + k_4\theta = M$$

or $$\Sigma k . \theta = M$$

where $\Sigma k = k_1 + k_2 + k_3 + k_4$

or $$\theta = \frac{M}{\Sigma k}$$

Hence $$m_1 = k_1\theta = k_1 \frac{M}{\Sigma k} = \frac{k_1}{\Sigma k} . M$$

Similarly, $$m_2 = \frac{k_2}{\Sigma k} . M$$

$$m_3 = \frac{k_3}{\Sigma k} . M$$

$$m_4 = \frac{k_4}{\Sigma k} . M$$

Factors $\frac{k_1}{\Sigma k}$, $\frac{k_2}{\Sigma k}$ etc are called the distribution factors of the members OA, OB etc.

Please see:

$$k_1 = 4E_1I_1/l_1$$

$$k_2 = 3E_2I_2/l_2$$

$$k_3 = 3E_3I_3/l_3$$

$$k_4 = 4E_4I_4/l_4$$