

SOLUTIONS TO THE PROBLEMS
in
TRANSPORT PHENOMENA
Second Edition
(2002)

by

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1A.1 Estimation of dense-gas viscosity.

a. Table E.1 gives $T_c = 126.2$ K, $p_c = 33.5$ atm, and $\mu_c = 180 \times 10^{-6}$ g/cm·s for N₂. The reduced conditions for the viscosity estimation are then:

$$p_r = p/p_c = (1000 + 14.7)/33.5 \times 14.7 = 2.06$$

$$T_r = T/T_c = (273.15 + (68 - 32)/1.8)/126.2 = 2.32$$

At this reduced state, Fig. 1.3-1 gives $\mu_r = 1.15$. Hence, the predicted viscosity is $\mu = \mu_r/\mu_c = 1.15 \times 180 \times 10^{-6} = 2.07 \times 10^{-4}$ g/cm·s. This result is then converted into the requested units by use of Table F.3-4:

$$\mu = 2.07 \times 10^{-4} \times 6.7197 \times 10^{-2} = 1.4 \times 10^{-4} \text{ lb}_m/\text{ft}\cdot\text{s}$$

1A.2 Estimation of the viscosity of methyl fluoride.

a. CH_3F has $M = 16.04 - 1.008 + 19.00 = 34.03$ g/g-mole, $T_c = 4.55 + 273.15 = 277.70$ K, $p_c = 58.0$ atm, and $\tilde{V}_c = 34.03/0.300 = 113.4$ cm³/g-mole. The critical viscosity is then estimated as

$$\mu_c = 61.6(34.03 \times 277.70)^{1/2}(113.4)^{-2/3} = 255.6 \text{ micropoise}$$

from Eq. 1.3-1a, and

$$\mu_c = 7.70(34.03)^{1/2}(58.0)^{2/3}(277.7)^{-1/6} = 263.5 \text{ micropoise}$$

from Eq. 1.3-1b.

The reduced conditions for the viscosity estimate are $T_r = (370 + 273.15)/277.70 = 2.32$, $p_r = 120/58.0 = 2.07$, and the predicted μ_r from Fig. 1.3-1 is 1.1. The resulting predicted viscosity is

$$\begin{aligned} \mu = \mu_r \mu_c &= 1.1 \times 255.6 \times 10^{-6} = 2.8 \times 10^{-4} \text{ g/cm}\cdot\text{s via Eq.1.3-1a, or} \\ &1.1 \times 263.5 \times 10^{-6} = 2.9 \times 10^{-4} \text{ g/cm}\cdot\text{s via Eq.1.3-1b.} \end{aligned}$$

1A.3 Computation of the viscosities of gases at low density.

Equation 1.4-14, with molecular parameters from Table E.1 and collision integrals from Table E.2, gives the following results:

For O_2 : $M = 32.00$, $\sigma = 3.433 \text{ \AA}$, $\varepsilon/\text{K} = 113 \text{ K}$. Then at 20°C , $kT/\varepsilon = 293.15/113 = 2.594$ and $\Omega_\mu = 1.086$. Equation 1.4-14 then gives

$$\begin{aligned}\mu &= 2.6693 \times 10^{-5} \frac{\sqrt{32.00 \times 293.15}}{(3.433)^2 \times 1.086} \\ &= 2.02 \times 10^{-4} \text{ g/cm}\cdot\text{s} \\ &= 2.02 \times 10^{-5} \text{ Pa}\cdot\text{s} \\ &= 2.02 \times 10^{-2} \text{ mPa}\cdot\text{s}.\end{aligned}$$

The reported value in Table 1.1-3 is $2.04 \times 10^{-2} \text{ mPa}\cdot\text{s}$.

For N_2 : $M = 28.01$, $\sigma = 3.667 \text{ \AA}$, $\varepsilon/\text{K} = 99.8 \text{ K}$. Then at 20°C , $kT/\varepsilon = 293.15/99.8 = 2.937$ and $\Omega_\mu = 1.0447$. Equation 1.4-14 then gives

$$\begin{aligned}\mu &= 2.6693 \times 10^{-5} \frac{\sqrt{28.01 \times 293.15}}{(3.667)^2 \times 1.0447} \\ &= 1.72 \times 10^{-4} \text{ g/cm}\cdot\text{s} \\ &= 1.72 \times 10^{-5} \text{ Pa}\cdot\text{s} \\ &= 1.72 \times 10^{-2} \text{ mPa}\cdot\text{s}.\end{aligned}$$

The reported value in Table 1.1-3 is $1.75 \times 10^{-2} \text{ mPa}\cdot\text{s}$.

For CH_4 , $M = 16.04$, $\sigma = 3.780 \text{ \AA}$, $\varepsilon/\text{K} = 154 \text{ K}$. Then at 20°C , $kT/\varepsilon = 293.15/154 = 1.904$ and $\Omega_\mu = 1.197$. Equation 1.4-14 then gives

$$\begin{aligned}\mu &= 2.6693 \times 10^{-5} \frac{\sqrt{16.04 \times 293.15}}{(3.780)^2 \times 1.197} \\ &= 1.07 \times 10^{-4} \text{ g/cm}\cdot\text{s} \\ &= 1.07 \times 10^{-5} \text{ Pa}\cdot\text{s} \\ &= 1.07 \times 10^{-2} \text{ mPa}\cdot\text{s}.\end{aligned}$$

The reported value in Table 1.1-3 is $1.09 \times 10^{-2} \text{ mPa}\cdot\text{s}$.

1A.4 Gas-mixture viscosities at low density.

The data for this problem are as follows:

Component	M	μ , poise $\times 10^6$
1(H ₂)	2.016	88.4
2(CCl ₂ F ₂)	120.92	124.0

Insertion of these data into Eq. 1.4-16 gives the following coefficients for mixtures of H₂ and Freon-12 at this temperature:

$$\Phi_{11} = \Phi_{22} = 1.0$$

$$\Phi_{12} = \frac{1}{\sqrt{8}} \left(1 + \frac{2.016}{120.92} \right)^{-1/2} \left[1 + \left(\frac{88.4}{124.0} \right)^{1/2} \left(\frac{120.92}{2.016} \right)^{1/4} \right]^2$$

$$= 3.934$$

$$\Phi_{21} = \frac{1}{\sqrt{8}} \left(1 + \frac{120.92}{2.016} \right)^{-1/2} \left[1 + \left(\frac{124.0}{88.4} \right)^{1/2} \left(\frac{2.016}{120.92} \right)^{1/4} \right]^2$$

$$= 0.0920$$

Equation 1.4-15 then gives the predicted mixture viscosities:

$x_1 =$ $1 - x_2$	$\sum_1 =$ $\sum x_\beta \Phi_{1\beta}$	$\sum_2 =$ $\sum x_\beta \Phi_{2\beta}$	$A :=$ $x_1 \mu_1 / \sum_1$	$B :=$ $x_2 \mu_2 / \sum_2$	$A + B =$ $\mu_{\text{mix}} \times 10^6$	$\mu_{\text{obs,poise}} \times 10^6$
0.00	3.934	1.000	0.0	124.0	(124.0)	124.0
0.25	3.200	0.773	6.9	120.3	127.2	128.1
0.50	2.467	0.546	18.1	113.6	131.7	131.9
0.75	1.734	0.319	38.2	97.2	135.4	135.1
1.00	1.000	0.092	88.4	0.0	(88.4)	88.4

1A.5 Viscosities of chlorine-air mixtures at low density.

Equation 1.4-14 and Tables E.1, E.2 give the following viscosities at 75°F (= 273.15 + (75 - 32)/1.8 = 297.03 K) and 1 atm:

For component 1, (Cl₂), $M_1 = 70.91$, $\sigma_1 = 4.115 \text{ \AA}$, $\varepsilon_1/K = 357\text{K}$; hence, $\kappa T/\varepsilon_1 = 297.03/357 = 0.832$ and $\Omega_{\mu,1} = 1.754$, and

$$\mu_1 = 2.6693 \times 10^{-5} \frac{\sqrt{70.91 \times 297.03}}{(4.115)^2 \times 1.754} = 1.304 \times 10^{-4} \text{ g/cm}\cdot\text{s} = 0.01304 \text{ cp.}$$

For component 2, (air), $M_2 = 28.97$, $\sigma = 3.617 \text{ \AA}$, $\varepsilon_1/K = 97.0\text{K}$; hence, $\kappa T/\varepsilon_1 = 297.03/97.0 = 3.062$ and $\Omega_{\mu,1} = 1.033$, and

$$\mu_2 = 2.6693 \times 10^{-5} \frac{\sqrt{28.97 \times 297.03}}{(3.617)^2 \times 1.033} = 1.832 \times 10^{-4} \text{ g/cm}\cdot\text{s} = 0.01832 \text{ cp.}$$

Eq. 1.4-16 then gives the following coefficients for Eq. 1.4-15 at this temperature:

$$\Phi_{11} = \Phi_{22} = 1.0$$

$$\begin{aligned} \Phi_{12} &= \frac{1}{\sqrt{8}} \left(1 + \frac{70.91}{28.97}\right)^{-1/2} \left[1 + \left(\frac{0.01304}{0.01832}\right)^{1/2} \left(\frac{28.97}{70.91}\right)^{1/4}\right]^2 \\ &= 0.5339 \end{aligned}$$

$$\begin{aligned} \Phi_{21} &= \frac{1}{\sqrt{8}} \left(1 + \frac{28.97}{70.91}\right)^{-1/2} \left[1 + \left(\frac{0.01832}{0.01304}\right)^{1/2} \left(\frac{70.91}{28.97}\right)^{1/4}\right]^2 \\ &= 1.8360 \end{aligned}$$

Equation 1.4-15 then gives the predicted mixture viscosities:

$x_1 =$ $1 - x_2$	$\sum_1 =$ $\sum x_\beta \Phi_{1\beta}$	$\sum_2 =$ $\sum x_\beta \Phi_{2\beta}$	$A :=$ $x_1 \mu_1 / \sum_1$	$B :=$ $x_2 \mu_2 / \sum_2$	$A + B =$ $\mu_{\text{mix, cp.}} \times 10^6$
0.00	0.5339	1.000	0.0	0.01832	0.0183
0.25	0.6504	1.2090	0.005012	0.011365	0.0164
0.50	0.7670	1.4180	0.008501	0.006460	0.0150
0.75	0.8835	1.6270	0.011070	0.002815	0.0139
1.00	1.000	1.8360	0.01304	0.0	0.0130

1A.6 Estimation of liquid viscosity.

a. The calculated values for Eq. 1.5-9 at 0°C and 100°C are as follows:

T, K	273.15	373.15
$\rho, g/cm^3$	0.9998	0.9584
$\tilde{V} = M/\rho, cm^3/g\text{-mole}$	18.01	18.80
$\Delta\tilde{U}_{vap, T_b}, cal/g\text{-mole} = 897.5 \times 18.016 \times 252.16/453.59$	8989.	8989.
$\Delta\tilde{U}_{vap, T_b}/RT = 8989/1.98721/T$	16.560	12.120
$\exp 0.408\Delta\tilde{U}_{vap, T_b}/RT$	859.6	140.5
$\tilde{N}h/\tilde{V}, g/cm\cdot s$	2.22×10^{-4}	2.12×10^{-4}
Predicted liquid viscosity, g/cm·s	0.19	0.0298

b. The predicted values for Eq. 1.5-11 at 0°C and 100°C are:

T, K	273.15	373.15
$\tilde{N}h/\tilde{V}, g/cm\cdot s$	2.22×10^{-4}	2.12×10^{-4}
$\exp(3.8T_b/T)$	179.7	44.70
Predicted liquid viscosity, g/cm·s	0.0398	0.0095

Summary of results:

Temperature, °C	0	100
Observed viscosity, centipoise[=]g/cm·s×100	1.787	0.2821
Prediction of Eq. 1.5-9	19.	2.98
Prediction of Eq. 1.5-11	3.98	0.95

Both equations give poor predictions. This is not surprising, since the empirical formulas in Eqs. 1.5-8 *et seq.* are inaccurate for water and for other associated liquids.

1A.7 Molecular velocity and mean free path.

From eq. 1.4-1, the mean molecular velocity in O₂ at 273.2 K is

$$\bar{u} = \sqrt{\frac{8RT}{\pi M}} = \sqrt{\frac{8 \times 8.31451 \times 10^7 \times 273.2}{\pi \times 32.00}} = 4.25 \times 10^4 \text{ cm/s}$$

From eq. 1.4-3, the mean free path in O₂ at 1 atm and 273.2 K is

$$\lambda = \frac{RT}{\sqrt{2}\pi d^2 p \tilde{N}} = \frac{82.0578 \times 273.2}{\sqrt{2}\pi(3 \times 10^{-8})^2 \times 1 \times 6.02214 \times 10^{23}} = 9.3 \times 10^{-6} \text{ cm}$$

Hence, the ratio of the mean free path to the molecular diameter is $(9.3 \times 10^{-6} / 3 \times 10^{-8}) = 3.1 \times 10^4$ under these conditions. At liquid states, on the other hand, the corresponding ratio would be on the order of unity or even less.

1B.1 Velocity profiles and stress components

a. $\tau_{xy} = \tau_{yx} = -\mu b$, and all other τ_{ij} are zero.

$\rho v_x v_x = \rho b^2 y^2$, and all other $\rho v_i v_j$ are zero.

b. $\tau_{xy} = \tau_{yx} = -2\mu b$, and all other τ_{ij} are zero.

$\rho v_x v_x = \rho b^2 y^2$, $\rho v_x v_y = \rho v_y v_x = \rho b^2 xy$, $\rho v_y v_y = \rho b^2 x^2$, and all other $\rho v_i v_j$ are zero.

c. All τ_{ij} are zero

$\rho v_x v_x = \rho b^2 y^2$, $\rho v_x v_z = \rho v_z v_x = -\rho b^2 xz$, $\rho v_z v_z = \rho b^2 x^2$.

components of $\rho \mathbf{v} \mathbf{v}$ may be given in the matrix:

$$\rho \mathbf{v} \mathbf{v} = \begin{pmatrix} \rho v_x v_x = \frac{1}{4} \rho b^2 x^2 & \rho v_x v_y = \frac{1}{4} \rho b^2 xy & \rho v_x v_z = -\frac{1}{2} \rho b^2 xz \\ \rho v_y v_x = \frac{1}{4} \rho b^2 xy & \rho v_y v_y = \frac{1}{4} \rho b^2 y^2 & \rho v_y v_z = -\frac{1}{2} \rho b^2 yz \\ \rho v_z v_x = -\frac{1}{2} \rho b^2 xz & \rho v_z v_y = -\frac{1}{2} \rho b^2 yz & \rho v_z v_z = \rho b^2 z^2 \end{pmatrix}$$

1B.2 A fluid in a state of rigid rotation

a. A particle within a rigid body rotating with an angular velocity vector \mathbf{w} has a velocity given by $\mathbf{v} = [\mathbf{w} \times \mathbf{r}]$. If the angular velocity vector is in the $+z$ -direction, then there are two nonzero velocity components given by $v_x = -w_z y$ and $v_y = +w_z x$. Hence the magnitude of the angular velocity vector is b in Problem 1B.1(c).

b. For the velocity components of Problem 1B.1(c),

$$\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} = 0 \quad \text{and} \quad \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 2b$$

c. In Eq. 1.2-4, we selected only the linear *symmetric* combinations of derivatives of the velocity, so that in pure rotation there would be no viscous forces present. In (b) we see that the antisymmetric combination is nonzero in a purely rotational motion.

1B.3 Viscosity of suspensions

Expanding the Mooney expression, we get (with $\varepsilon = \phi/\phi_0$)

$$\begin{aligned}\frac{\mu_{\text{eff}}}{\mu_0} &= 1 + \left(\frac{\frac{5}{2}\phi}{1-\varepsilon}\right) + \frac{1}{2!}\left(\frac{\frac{5}{2}\phi}{1-\varepsilon}\right)^2 + \frac{1}{3!}\left(\frac{\frac{5}{2}\phi}{1-\varepsilon}\right)^3 + \dots \\ &= 1 + \frac{5}{2}\phi(1 + \varepsilon + \varepsilon^2 + \dots) + \frac{25}{8}\phi^2(1 + 2\varepsilon + \dots) + \frac{125}{48}\phi^3(1 + \dots) + \dots \\ &= 1 + \frac{5}{2}\phi + \phi^2\left(\frac{25}{8} + \frac{5}{2}\frac{1}{\phi_0}\right) + \phi^3\left(\frac{125}{48} + \frac{25}{4}\frac{1}{\phi_0} + \frac{5}{2}\frac{1}{\phi_0^2}\right) + \dots\end{aligned}$$

The first two terms match exactly with the first two terms in Eq. 1B.3-1. We can make the third term match exactly, by setting

$$\frac{25}{8} + \frac{5}{2}\frac{1}{\phi_0} = 7.17 \quad \text{whence } \phi_0 = 0.618$$

and the coefficient of ϕ^3 becomes

$$\frac{125}{48} + \frac{25}{4}\frac{1}{0.618} + \frac{5}{2}\frac{1}{0.382} = 20.26$$

If we try $\phi_0 = 0.70$, the coefficients of ϕ^2 and ϕ^3 become 6.70 and 17.6 respectively. This gives a somewhat better fit of Vand's data.

1C.1 Some consequences of the Maxwell-Boltzmann equation

a. The mean speed is

$$\bar{u} = \frac{\int_0^{\infty} u^3 e^{-mu^2/2kT} du}{\int_0^{\infty} u^2 e^{-mu^2/2kT} du} = \sqrt{\frac{2kT}{m}} \frac{\int_0^{\infty} \xi^3 e^{-\xi^2} d\xi}{\int_0^{\infty} \xi^2 e^{-\xi^2} d\xi} = \sqrt{\frac{2kT}{m}} \frac{\frac{1}{2}}{\frac{1}{4}\sqrt{\pi}} = \sqrt{\frac{8kT}{\pi m}}$$

b. First rewrite Eq. 1C.1-4 as

$$\bar{u}_x = \frac{\int_{-\infty}^{\infty} u_x e^{-mu_x^2/2kT} du_x}{\int_{-\infty}^{\infty} e^{-mu_x^2/2kT} du_x} \cdot \frac{\int_{-\infty}^{\infty} e^{-mu_y^2/2kT} du_y}{\int_{-\infty}^{\infty} e^{-mu_y^2/2kT} du_y} \cdot \frac{\int_{-\infty}^{\infty} e^{-mu_z^2/2kT} du_z}{\int_{-\infty}^{\infty} e^{-mu_z^2/2kT} du_z}$$

The integral over u_x in the numerator of the first factor is zero because the integrand is the product of a factor " u_x " (an odd function of the integration variable about $u_x = 0$) and an exponential function (an even function), and the range of integration extends equally far in the positive and negative directions.

c. The mean kinetic energy per molecule is

$$\frac{1}{2} m \overline{u^2} = \frac{1}{2} m \frac{\int_0^{\infty} u^4 e^{-mu^2/2kT} du}{\int_0^{\infty} u^2 e^{-mu^2/2kT} du} = \frac{1}{2} m \frac{2kT}{m} \frac{\int_0^{\infty} \xi^4 e^{-\xi^2} d\xi}{\int_0^{\infty} \xi^2 e^{-\xi^2} d\xi} = kT \frac{\frac{3}{8}\sqrt{\pi}}{\frac{1}{4}\sqrt{\pi}} = \frac{3}{2} kT$$

and is thus $\frac{1}{2} kT$ for each degree of freedom.

1C.2 The wall collision frequency

When we change to dimensionless variables in the second line of Eq. 1C.2-1, we get

$$\begin{aligned} Z &= n \left(\frac{m}{2\pi kT} \right)^{3/2} \frac{2kT}{m} \left(\int_0^\infty \xi e^{-\xi^2} d\xi \right) \sqrt{\frac{2kT}{m}} \left(\int_0^\infty e^{-\xi^2} d\xi \right) \sqrt{\frac{2kT}{m}} \left(\int_0^\infty e^{-\xi^2} d\xi \right) \\ &= n \left(\frac{m}{2\pi kT} \right)^{3/2} \left(\frac{2kT}{m} \cdot \frac{1}{2} \right) \left(\sqrt{\frac{2kT}{m}} \cdot \sqrt{\pi} \right) \left(\sqrt{\frac{2kT}{m}} \cdot \sqrt{\pi} \right) = n \sqrt{\frac{kT}{2\pi m}} \end{aligned}$$

1C.3 The pressure in an ideal gas

a. The dimensions of the quantities in Eq. 1C.3-1 are

$$\begin{array}{lll} S & [=] & L^2 \\ u_x & [=] & L/t \\ \Delta t & [=] & t \\ m & [=] & M \\ f & [=] & (1/L^3)(L/t)^{-3} \\ du_x du_y du_z & [=] & (L/t)^3 \end{array}$$

Using these units, one finds that the expression on the right of Eq. 1C.3-1 has units of M/Lt^2 (which are the same as the units of force per area).

b. Combining Eqs. 1C.1-1 and 1C.3-1 we get

$$\begin{aligned} p &= 2nm \left(\frac{m}{2\pi kT} \right)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} u_x^2 e^{-m(u_x^2 + u_y^2 + u_z^2)/2kT} du_x du_y du_z \\ &= 2nm \left(\frac{m}{2\pi kT} \right)^{3/2} \int_{-\infty}^{\infty} u_x^2 e^{-mu_x^2/2kT} du_x \cdot \int_{-\infty}^{\infty} e^{-mu_y^2/2kT} du_y \cdot \int_0^{\infty} e^{-mu_z^2/2kT} du_z \\ &= 2nm \left(\frac{2kT}{m} \right) \left(\frac{1}{\pi} \right)^{3/2} \int_0^{\infty} \xi^2 e^{-\xi^2} d\xi \cdot \int_{-\infty}^{\infty} \eta e^{-\eta^2} d\eta \cdot \int_{-\infty}^{\infty} \zeta^2 e^{-\zeta^2} d\zeta \\ &= 2nm \left(\frac{2kT}{m} \right) \left(\frac{1}{\pi} \right)^{3/2} \left(\frac{\sqrt{\pi}}{4} \right) (\sqrt{\pi}) (\sqrt{\pi}) = nkT \end{aligned}$$

1D.1 Uniform rotation of a fluid

a. For the special case that $\mathbf{w} = \delta_z w$, we get

$$\mathbf{v} = [\mathbf{w} \times \mathbf{r}] = \sum_i \sum_j \sum_k \varepsilon_{ijk} \delta_i w_j x_k = w(\delta_1 \varepsilon_{132} y + \delta_2 \varepsilon_{231} x) = w(-\delta_1 y + \delta_2 x)$$

Then using Eqs. A.6-1, 2, 13 and 14, we can get the velocity components in cylindrical coordinates

$$\begin{aligned} v_r &= (\mathbf{v} \cdot \delta_r) = w((- \delta_x y + \delta_y x) \cdot \delta_r) = w(-y \cos \theta + x \sin \theta) \\ &= w(-r \sin \theta \cos \theta + x \cos \theta \sin \theta) = 0 \end{aligned}$$

$$\begin{aligned} v_\theta &= (\mathbf{v} \cdot \delta_\theta) = w((- \delta_x y + \delta_y x) \cdot \delta_\theta) = w((-y)(-\sin \theta) + x \cos \theta) \\ &= w(r \sin \theta \sin \theta + r \cos \theta \cos \theta) = wr \end{aligned}$$

Therefore, the angular velocity of every point in the fluid is $v_\theta/r = w$, which is a constant, and there is no radial velocity. This is the way a rigid body rotates at constant angular velocity.

b. The vector operations are (using the abbreviated notation of §A.9 and the Einstein summation convention)

$$(\nabla \cdot \mathbf{v}) = \partial_i v_i = \partial_i \varepsilon_{imn} w_m x_n = \varepsilon_{imn} w_m \delta_{in} = \varepsilon_{nmn} w_m = 0$$

$$\begin{aligned} \{\nabla \mathbf{v}\}_{ij} &= \{\nabla[\mathbf{w} \times \mathbf{r}]\}_{ij} = \partial_i \varepsilon_{jmn} w_m x_n = \varepsilon_{jmn} w_m \delta_{in} = \varepsilon_{jmi} w_m \\ &= -\{\nabla \mathbf{v}\}_{ji} = -\{\nabla \mathbf{v}\}^{\dagger} \end{aligned}$$

1D.2 Force on a surface of arbitrary orientation.

a. We can specify the surface area *and* the orientation of the surface of ΔOBC as $\mathbf{n}dS$. To project this surface onto the yz -plane, we take the dot product with δ_x , so that the area of ΔOBC is $(\mathbf{n} \cdot \delta_x)dS$.

b. The force per unit area on three triangles perpendicular to the three coordinate axes are

$$\text{Force on } \Delta OBC = \delta_x \pi_{xx} + \delta_y \pi_{xy} + \delta_z \pi_{xz}$$

$$\text{Force on } \Delta OCA = \delta_x \pi_{yx} + \delta_y \pi_{yy} + \delta_z \pi_{yz}$$

$$\text{Force on } \Delta OAB = \delta_x \pi_{zx} + \delta_y \pi_{zy} + \delta_z \pi_{zz}$$

c. Force balance on the volume OABC is then

$$\begin{aligned} \pi_n dS &= (\delta_x \pi_{xx} + \delta_y \pi_{xy} + \delta_z \pi_{xz})(\mathbf{n} \cdot \delta_x) dS \\ &\quad + (\delta_x \pi_{yx} + \delta_y \pi_{yy} + \delta_z \pi_{yz})(\mathbf{n} \cdot \delta_y) dS \\ &\quad + (\delta_x \pi_{zx} + \delta_y \pi_{zy} + \delta_z \pi_{zz})(\mathbf{n} \cdot \delta_z) dS \end{aligned}$$

or

$$\begin{aligned} \pi_n &= [\mathbf{n} \cdot \delta_x \delta_x \pi_{xx}] + [\mathbf{n} \cdot \delta_x \delta_y \pi_{xy}] + [\mathbf{n} \cdot \delta_x \delta_z \pi_{xz}] \\ &\quad + [\mathbf{n} \cdot \delta_y \delta_x \pi_{yx}] + [\mathbf{n} \cdot \delta_y \delta_y \pi_{yy}] + [\mathbf{n} \cdot \delta_y \delta_z \pi_{yz}] \\ &\quad + [\mathbf{n} \cdot \delta_z \delta_x \pi_{zx}] + [\mathbf{n} \cdot \delta_z \delta_y \pi_{zy}] + [\mathbf{n} \cdot \delta_z \delta_z \pi_{zz}] \\ &= \sum_i \sum_j [\mathbf{n} \cdot \delta_i \delta_j \pi_{ij}] = [\mathbf{n} \cdot \boldsymbol{\pi}] \end{aligned}$$

2A.1 Thickness of a falling film.

a. The volume flow rate w/ρ per unit wall width W is obtained from Eq. 2.2-25:

$$\frac{w}{\rho W} = \frac{\nu \text{Re}}{4} = \frac{(1.0037 \times 10^{-2})(10)}{4} = 2.509 \times 10^{-2} \text{ cm}^2/\text{s}$$

Here the kinematic viscosity ν for liquid water at 20°C was obtained from Table 1.1-2. Since 1 ft = 12 × 2.54 cm, 1 hr = 3600 s, and 1 gal = 231.00 in³ × (2.54 cm/in)³ = 3785.4 cm³ (see Appendix F), the result in the requested units is

$$\begin{aligned} \frac{w}{\rho W} &= 0.02509 \text{ cm}^2/\text{s} \times \frac{1}{3785.4} \text{ gal/cm}^2 \times 30.48 \text{ cm/ft} \times 3600 \text{ s/hr} \\ &= 0.727 \text{ U.S. gal/hr}\cdot\text{ft} \end{aligned}$$

b. The film thickness is calculated from Eqs. 2.2-25 and 2.2-22 as

$$\begin{aligned} \delta &= \left(\frac{3\nu}{g \cos \beta} \frac{w}{\rho W} \right)^{1/3} \\ &= \left(\frac{3\nu}{g \cos \beta} \frac{\nu \text{Re}}{4} \right)^{1/3} \\ &= \left(\frac{3 \times 1.0037 \times 10^{-2}}{(980.665)(1.0)} (2.509 \times 10^{-2}) \right)^{1/3} = 0.009167 \text{ cm} \\ &= 0.00361 \text{ in.} \end{aligned}$$

2A.2 Determination of capillary radius by flow measurement.

Assuming the flow to be laminar, we solve Eq. 2.3-21 for the capillary radius:

$$R = \sqrt[4]{\frac{8\mu Lw}{\pi\rho\Delta\mathcal{P}}} = \sqrt[4]{\frac{8\nu Lw}{\pi\Delta\mathcal{P}}}$$

Insertion of the data in mks units gives

$$\begin{aligned} R &= \sqrt[4]{\frac{8(4.03 \times 10^{-5})(0.5002)(2.997 \times 10^{-3})}{(3.1416)(4.829 \times 10^5)}} \\ &= \sqrt[4]{3.186 \times 10^{-13}} \\ &= 7.51 \times 10^{-4} \text{ m} = 7.51 \times 10^{-2} \text{ cm} \end{aligned}$$

As a check on this result, we calculate the corresponding Reynolds number:

$$\begin{aligned} \text{Re} &= \frac{D\langle v_z \rangle \rho}{\mu} = \frac{4w}{\pi D\mu} = \frac{2w}{\pi R\nu\rho} \\ &= \frac{2}{\pi (7.51 \times 10^{-4})(4.03 \times 10^{-5})(0.9552 \times 10^3)} = 66.0 \end{aligned}$$

This value supports our assumption of laminar flow. Since the entrance length, $L_e = 0.035D\text{Re} = 0.35 \text{ cm}$ is less than L , the entrance-effect correction to R is at most of the order of $|[1 - (L_e/L)]^{1/4} - 1|$, or 0.2 percent of R in the present example.

Difficulties with this method include: (1) Inability to account for departures from a straight, circular cylindrical wall geometry. (2) Inability to account for inadvertent spatial and temporal variations of temperature, hence of the fluid density and viscosity.

A simpler method is to measure the length L and mass m of a small slug of liquid mercury (or another liquid of known density) injected into the tube, and calculate the mean radius R of the slug as $(m/[\rho\pi L])^{1/2}$, on the assumption that the slug is a right circular cylinder. This method allows comparisons of mean R values for various intervals of the tube length.

2A.3 Volume rate of flow through an annulus.

Assuming the flow to be laminar, we use Eq. 2.4-17 to calculate the volume flow rate w/ρ , with the specifications

$$\kappa = 0.495/1.1 = 0.45$$

$$\mu = 136.8 \text{ (lb}_m\text{/ft}\cdot\text{hr)}(1 \text{ hr}/3600\text{s}) = 3.80 \times 10^{-2} \text{ lb}_m\text{/ft}\cdot\text{s}$$

$$(\mathcal{P}_0 - \mathcal{P}_L) = (5.39 \text{ psi})(4.6330 \times 10^3 \text{ poundals/ft}^2\text{/psi}) = 2.497 \times 10^4 \text{ lb}_m\text{/ft}\cdot\text{s}^2$$

$$R = 1.1 \text{ in.} = 1.1/12 \text{ ft}$$

Here Appendix F has been used for the conversions of units. With these specifications, Eq. 2.4-17 gives

$$\begin{aligned} \frac{w}{\rho} &= \frac{(\pi)(2.497 \times 10^4)(1.1/12)^4}{(8)(3.80 \times 10^{-2})(27)} \left[(1 - (0.45)^4) - \frac{(1 - (0.45)^2)^2}{\ln(1/0.45)} \right] \\ &= (0.49242) \left[(1 - 0.04101) - \frac{(1 - 0.2025)^2}{\ln(1/0.495)} \right] \\ &= (0.6748)[0.1625] = 0.110 \text{ ft}^3/\text{s} \end{aligned}$$

As a check on our assumption of laminar flow, we calculate the Reynolds number:

$$\begin{aligned} \text{Re} &= \frac{2R(1 - \kappa)\langle v_z \rangle \rho}{\mu} = \frac{2w}{\pi R \mu (1 + \kappa)} \\ &= \frac{2(0.110)(80.3)}{(3.1416)(1.1/12)(3.80 \times 10^{-2})(1.45)} = 1110 \end{aligned}$$

This value is well within the laminar range, so our assumption of laminar flow is confirmed.

2A.4 Loss of catalyst particles in stack gas.

a. Rearrangement of Eq. 2.6-17 gives the terminal velocity

$$v_t = D^4(\rho_s - \rho)g/18\mu$$

in which D is the sphere diameter. Particles settling at v_t greater than the centerline gas velocity will not go up the stack. Hence, the value of D that corresponds to $v_t = 1.0$ ft/s will be the maximum diameter of particles that can be lost in the stack gas of the present system.

Conversions of data to cgs units give

$$\begin{aligned}v_t &= (1 \text{ ft/s})(12 \times 2.54 \text{ cm/ft}) = 30.48 \text{ cm/s} \\ \rho &= (0.045 \text{ lb}_m/\text{ft}^3)(453.59 \text{ g/lb}_m)((12 \times 2.54)^{-3} \text{ ft}^3/\text{cm}^3) \\ &= 7.2 \times 10^{-4} \text{ g/cm}^3\end{aligned}$$

Hence,

$$\begin{aligned}D_{\max} &= \sqrt{\frac{18\mu v_t}{(\rho_s - \rho)g}} = \sqrt{\frac{(18)(0.00026)(30.48)}{(1.2 - 7.2 \times 10^{-4})(980.7)}} \\ &= \sqrt{1.21 \times 10^{-4}} = 1.1 \times 10^{-2} \text{ cm} = 110 \text{ microns}\end{aligned}$$

b. Equation 2.6-17 was derived for $Re \ll 1$, but holds approximately up to $Re=1$. For the system at hand,

$$Re = \frac{Dv_t\rho}{\mu} = \frac{(1.1 \times 10^{-2})(30.5)(7.2 \times 10^{-4})}{(0.00026)} = 0.93$$

Hence, the result in *a.* is approximately correct. Methods are given in Chapter 6 for solving problems of this type without the creeping-flow assumption.

2B.1 Different choice of coordinates for the falling film problem

Set up a momentum balance as before, and obtain the differential equation

$$\frac{d\tau_{\bar{x}z}}{d\bar{x}} = \rho g \cos \beta$$

Since no momentum is transferred at $\bar{x} = \delta$, then at that plane $\tau_{\bar{x}z} = 0$. This boundary condition enables us to find that $C_1 = -\rho g \delta \cos \beta$, and the momentum flux distribution is

$$\tau_{\bar{x}z} = -\rho g \delta \cos \beta \left(1 - \frac{\bar{x}}{\delta}\right)$$

Note that the momentum flux is in the negative \bar{x} -direction.

Insertion of Newton's law of viscosity $\tau_{\bar{x}z} = -\mu(dv_z/d\bar{x})$ into the foregoing equation gives the differential equation for the velocity distribution:

$$\frac{dv_z}{d\bar{x}} = \left(\frac{\rho g \delta \cos \beta}{\mu}\right) \left(1 - \frac{\bar{x}}{\delta}\right)$$

This first-order differential equation can be integrated to give

$$v_z = \left(\frac{\rho g \delta^2 \cos \beta}{\mu}\right) \left(\frac{\bar{x}}{\delta} - \frac{1}{2} \left(\frac{\bar{x}}{\delta}\right)^2\right)$$

The constant C_2 is zero, because $v_z = 0$ at $\bar{x} = 0$.

We note that \bar{x} and x are related by $\bar{x}/\delta = 1 - (x/\delta)$. When this is substituted into the velocity distribution above, we get

$$v_z = \left(\frac{\rho g \delta^2 \cos \beta}{\mu}\right) \left[\left(1 - \frac{x}{\delta}\right) - \frac{1}{2} \left[1 - 2\frac{x}{\delta} + \left(\frac{\bar{x}}{\delta}\right)^2\right]\right]$$

which can be rearranged to give Eq. 2.2-18.

2B.2 Alternate procedure for solving flow problems

Substituting Eq. 2.2-14 into Eq. 2.2-10 gives

$$\frac{d}{dx} \left(-\mu \frac{dv_z}{dx} \right) = \rho g \cos \beta \quad \text{or} \quad \frac{d^2 v_z}{dx^2} = -\frac{\rho g \cos \beta}{\mu}$$

Integrate twice with respect to x (see Eq. C.1-10) and get

$$v_z = -\frac{\rho g \cos \beta}{2\mu} x^2 + C_1 x + C_2$$

Then use the no-slip boundary condition that $v_z = 0$ at $x = \delta$, and the zero momentum flux boundary condition that $dv_z/dx = 0$ at $x = 0$. The second gives $C_1 = 0$, and the first gives $C_2 = (\rho g \cos \beta / 2\mu) \delta^2$. Substitution of these constants into the general solution and rearranging then gives Eq. 2.2-18.

2B.3 Laminar flow in a narrow slit

a. The momentum balance leads to

$$\frac{d}{dx} \tau_{xz} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L} \quad \text{and} \quad \tau_{xz} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L} x + C_1$$

Substitution of Newton's law $\tau_{xz} = -\mu \frac{dv_z}{dx}$ into the above gives

$$\frac{dv_z}{dx} = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{\mu L} x + \frac{C_1}{\mu} \quad \text{or} \quad v_z = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)x^2}{2\mu L} + \frac{C_1}{\mu} x + C_2.$$

Use of the no-slip boundary conditions at $x = \pm B$ gives the expressions in Eq. 2B.3-1 and 2. One can also see that $C_1 = 0$ directly, since we know that the velocity distribution must be symmetrical about the plane $x = 0$.

b. The maximum velocity is at the middle of the slit and is

$$v_{z,\max} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{2\mu L} \quad \text{and hence} \quad \frac{v_z}{v_{z,\max}} = 1 - \left(\frac{x}{B}\right)^2$$

The ratio of the average to the maximum velocity is then

$$\frac{\langle v_z \rangle}{v_{z,\max}} = \frac{\int_0^W \int_{-B}^B [1 - (x/B)^2] dx dy}{\int_0^W \int_{-B}^B dx dy} = \frac{\int_0^1 (1 - \xi^2) d\xi}{\int_0^1 d\xi} = \left(1 - \frac{1}{3}\right) = \frac{2}{3}$$

c. The mass rate of flow is

$$w = \rho(2BW)\langle v_z \rangle = \rho(2BW)\left(\frac{2}{3}\right) \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{2\mu L} = \frac{2}{3} \frac{(\mathcal{P}_0 - \mathcal{P}_L)\rho B^3 W}{\mu L}$$

d. In Eq. 2.5-22, set both viscosities equal to μ . set b equal to B , and multiply by $BW\rho$.

2B.4 Laminar slit flow with a moving wall ("plane Couette flow")

Start with the velocity distribution from part (a) of Problem 2B.3 (in terms of the integration constants). Determine C_1 and C_2 from the boundary conditions that $v_z = 0$ at $x = -B$, and $v_z = V$ at $x = B$. This leads to

$$v_z = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{2\mu L} \left[1 - \left(\frac{x}{B} \right)^2 \right] + \frac{V}{2} \left(1 + \frac{x}{B} \right)$$

This expression can be differentiated with respect to x and then Newton's law of viscosity $\tau_{xz} = -\mu(dv_z/dx)$ can be used to get the expression for the stress tensor. Notice that the velocity distribution is no longer symmetric about the midplane, so that $C_1 \neq 0$.

2B.5 Interrelation of slit and annulus formulas

From Eq. 2.4-17 we get

$$\begin{aligned}w\left(\frac{8\mu L}{\pi(\mathfrak{P}_0 - \mathfrak{P}_L)R^4\rho}\right) &= \left(1 - (1 - \varepsilon)^4\right) + \frac{\left(1 - (1 - \varepsilon)^2\right)^2}{\ln(1 - \varepsilon)} \\&= \left(1 - 1 + 4\varepsilon - 6\varepsilon^2 + 4\varepsilon^3 - \varepsilon^4\right) + \frac{\left(1 - 1 + 2\varepsilon - \varepsilon^2\right)^2}{-\varepsilon - \frac{1}{2}\varepsilon^2 - \frac{1}{3}\varepsilon^3 - \frac{1}{4}\varepsilon^4 - \dots} \\&= \left(4\varepsilon - 6\varepsilon^2 + 4\varepsilon^3 - \varepsilon^4\right) - \frac{\left(4\varepsilon^2 - 4\varepsilon^3 + \varepsilon^4\right)}{\varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 + \frac{1}{4}\varepsilon^4 + \dots} \\&= \left(4\varepsilon - 6\varepsilon^2 + 4\varepsilon^3 - \varepsilon^4\right) - \left(4\varepsilon - 6\varepsilon^2 + \frac{8}{3}\varepsilon^3 - \frac{1}{3}\varepsilon^4 + \dots\right)\end{aligned}$$

This gives, finally, a result in agreement with Eq. 2B.5-1

$$w = \frac{\pi(\mathfrak{P}_0 - \mathfrak{P}_L)R^4\rho}{8\mu L} \left(\frac{4}{3}\varepsilon^3 - \frac{2}{3}\varepsilon^4 + \dots\right)$$

2B.6 Flow of a film on the outside of a circular tube

a. A momentum balance on the film gives

$$-\frac{d(r\tau_{rz})}{dr} + \rho gr = 0 \quad \text{or} \quad \mu \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) + \rho gr = 0$$

The latter may be integrated to give

$$v_z = -\frac{\rho gr^2}{4\mu} + C_1 \ln r + C_2$$

Next use the boundary conditions that at $r = R$, $v_z = 0$ (no slip) and that at $r = aR$, $dv_z/dr = 0$. When the integration constants have been found, we get for the velocity distribution

$$v_z = \frac{\rho g R^2}{4\mu} \left[1 - \left(\frac{r}{R} \right)^2 + 2a^2 \ln \frac{r}{R} \right]$$

b. The mass rate of flow in the film is then

$$w = \int_0^{2\pi} \int_R^{aR} \rho v_z r dr d\theta = 2\pi R^2 \rho \int_1^a v_z \xi d\xi$$

in which a dimensionless radial coordinate $\xi = r/R$ has been introduced. Then

$$\begin{aligned} w &= \frac{\pi \rho^2 g R^4}{2\mu} \int_1^a (1 - \xi^2 + 2a^2 \ln \xi) \xi d\xi \\ &= \frac{\pi \rho^2 g R^4}{2\mu} \left(\frac{1}{2} \xi^2 - \frac{1}{4} \xi^4 + 2a^2 \left[-\frac{1}{4} \xi^2 + \frac{1}{2} \xi^2 \ln \xi \right] \right) \Big|_1^a \\ &= \frac{\pi \rho^2 g R^4}{8\mu} (-1 + 4a^2 - 3a^4 + 4a^4 \ln a) \end{aligned}$$

c. If we set $a = 1 + \varepsilon$ (where ε is small) and expand in powers of ε using §C.2, we get

$$w = \frac{\pi\rho^2 gR^4}{8\mu} \left(\frac{16}{3} \varepsilon^3 + O(\varepsilon^4) \right) \approx \frac{2\pi\rho^2 gR^4 \varepsilon^3}{3\mu}$$

This is in agreement with Eq. 2.2-21 if we make the identifications $W = 2\pi R$ and $\delta = \varepsilon R$ (and furthermore consider only the case that $\cos\beta = 1$).

2B.7 Annular flow with inner cylinder moving axially

a. The momentum balance is the same as that in Eq. 2.3-11 or Eq. 2.4-2, but with the pressure-difference term omitted. We can substitute Newton's law of viscosity into this equation to get

$$-\mu \frac{dv_z}{dr} = \frac{C_1}{r}, \text{ whence } v_z = -\frac{C_1}{\mu} \ln r + C_2 \text{ or } \frac{v_z}{v_0} = -D_1 \ln \frac{r}{R} + D_2$$

That is, we select new integration constants, so that they are dimensionless. These integration constants are determined from the no-slip conditions at the cylindrical surfaces: $v_z(\kappa R) = v_0$ and $v_z(R) = 0$. The constants of integration are $D_2 = 0$ and $D_1 = -1/\ln \kappa$. This leads then directly to the result given in the book.

b. The mass rate of flow is

$$\begin{aligned} w &= \int_0^{2\pi} \int_{\kappa R}^R \rho v_z r dr d\theta = 2\pi \rho \frac{v_0 R^2}{\ln \kappa} \int_{\kappa}^1 (\ln \xi) \xi d\xi \\ &= 2\pi \rho \frac{v_0 R^2}{\ln \kappa} \left(\frac{1}{2} \xi^2 \ln \xi - \frac{1}{4} \xi^2 \right) \Big|_{\kappa}^1 = 2\pi \rho \frac{v_0 R^2}{\ln \kappa} \left(-\frac{1}{2} \kappa^2 \ln \kappa - \frac{1}{4} (1 - \kappa^2) \right) \end{aligned}$$

which is equivalent to the answer in the text.

c. The force on a length L of the rod

$$F = \int_0^L \int_0^{2\pi} \left(+\mu \frac{dv_z}{dr} \right) \Big|_{r=\kappa R} \kappa R d\theta dz = 2\pi \kappa R L \mu v_0 \frac{(1/\kappa R)}{\ln \kappa}$$

which gives the expression in the book.

d. When we replace κ by $1 - \varepsilon$ and expand in a Taylor series, we get

$$F = 2\pi L (-\mu) v_0 \frac{1}{-(\varepsilon + \frac{1}{2} \varepsilon^2 + \frac{1}{3} \varepsilon^3 + \frac{1}{4} \varepsilon^4 \dots)} = \frac{2\pi L \mu v_0}{\varepsilon} \left(1 - \frac{1}{2} \varepsilon - \frac{1}{12} \varepsilon^2 \dots \right)$$

To get this last result one has to do a long division involving the polynomial in the next-to-last step.

2B.8 Analysis of a capillary flowmeter

Designate the water by fluid "I" and the carbon tetrachloride by "II". Label the distance from B to C as "J". The mass rate of flow in the tube section "AB" is given by

$$w = \frac{\pi(\mathfrak{P}_A - \mathfrak{P}_B)R^4\rho_I}{8\mu L} = \frac{\pi[(p_A - p_B) + \rho_I gh]R^4\rho_I}{8\mu L}$$

Since the fluid in the manometer is not moving, the pressures at D and E must be equal; hence

$$p_A + \rho_I gh + \rho_I gJ + \rho_I gH = p_B + \rho_I gJ + \rho_{II} gH$$

from which we get

$$p_A - p_B + \rho_I gh = (\rho_{II} - \rho_I)gH$$

Insertion of this into the first equation above gives the expression for the mass rate of flow in terms of the difference in the densities of the two fluids, the acceleration of gravity, and the height H .

2B.9 Low-density phenomena in compressible tube flow

When we replace no-slip boundary condition of Eq. 2.3-17 by Eq. 2B.9-1, we get

$$C_2 = \frac{(p_0 - p_L)R^2}{4\mu L} + \frac{(p_0 - p_L)R\zeta}{2\mu L}$$

so that the velocity distribution in the tube is

$$v_z = \frac{(p_0 - p_L)R^2}{4\mu L} \left[1 - \left(\frac{r}{R} \right)^2 \right] + \frac{(p_0 - p_L)R\zeta}{2\mu L}$$

Next we write the expression for w , but consider only the flow through a length dz of the tube:

$$w = \int_0^{2\pi} \int_0^R \rho(z) v_z(r, z) r dr d\theta = 2\pi R^2 \left(\frac{pM}{R_g T} \right) \int_0^1 v_z \xi d\xi$$

where we have introduced the ideal gas law, with R_g being the gas constant (we use a subscript g here to distinguish the gas constant from the tube radius). We have also introduced a dimensionless radial coordinate. When we introduce the velocity distribution above, we get

$$\begin{aligned} w &= \frac{2\pi R^4}{4\mu} \left(\frac{M}{R_g T} \right) \int_0^1 \left(-p \frac{dp}{dz} \right) \left[(1 - \xi^2) + \frac{2\zeta_0}{Rp} \right] \xi d\xi \\ &= \frac{\pi R^4}{8\mu} \left(\frac{M}{R_g T} \right) \left(-p \frac{dp}{dz} \right) \left(1 + \frac{4\zeta_0}{Rp} \right) \end{aligned}$$

This is now integrated over the length of the tube, keeping mind that the mass flow rate w is constant over the entire length

$$\int_0^L w dz = -\frac{\pi R^4}{8\mu} \left(\frac{M}{R_g T} \right) \int_{p_0}^{p_L} \left(p + \frac{4\zeta_0}{R} \right) dp$$

$$= \frac{\pi R^4}{8\mu} \left(\frac{M}{R_g T} \right) \left(\frac{p_0^2 - p_L^2}{2} + \frac{4\zeta_0}{R} (p_0 - p_L) \right)$$

This gives

$$\begin{aligned} w &= \frac{\pi(p_0 - p_L)R^4}{8\mu L} \left(\frac{M}{R_g T} \right) \left(\frac{p_0 + p_L}{2} + \frac{4\zeta_0}{R} \right) \\ &= \frac{\pi(p_0 - p_L)R^4}{8\mu L} \left(\frac{p_{\text{avg}} M}{R_g T} \right) \left(1 + \frac{4\zeta_0}{R p_{\text{avg}}} \right) \end{aligned}$$

which leads then to Eq. 2B.9-2.

2B.10 Incompressible flow in a slightly tapered tube

a. The radius at any downstream distance is

$$R(z) = R_0 + (R_L - R_0)(z/L)$$

b. Changing the independent variable proceeds as follows:

$$w = \frac{\pi R^4 \rho}{8\mu} \left(-\frac{d\mathcal{P}}{dR} \right) \left(\frac{dR}{dz} \right) = \frac{\pi R^4 \rho}{8\mu} \left(-\frac{d\mathcal{P}}{dR} \right) \left(\frac{R_L - R_0}{L} \right)$$

c. First we rearrange the equation in (b) to get

$$-\frac{d\mathcal{P}}{dR} = \left(\frac{8\mu w}{\pi \rho} \right) \left(\frac{L}{R_L - R_0} \right) \frac{1}{R^4}$$

Then we integrate this equation to get

$$-\int_{\mathcal{P}_0}^{\mathcal{P}_L} d\mathcal{P} = \left(\frac{8\mu w}{\pi \rho} \right) \left(\frac{L}{R_L - R_0} \right) \int_{R_0}^{R_L} \frac{1}{R^4} dR$$

whence we can get the pressure difference in terms of the mass rate of flow

$$\mathcal{P}_0 - \mathcal{P}_L = \left(\frac{8\mu w L}{3\pi \rho} \right) \left(\frac{R_L^{-3} - R_0^{-3}}{R_0 - R_L} \right)$$

Next we solve to get the mass flow rate

$$w = \left(\frac{3\pi(\mathcal{P}_0 - \mathcal{P}_L)\rho}{8\mu L} \right) \left(\frac{R_0 - R_L}{R_L^{-3} - R_0^{-3}} \right) = \left(\frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)R_0^4 \rho}{8\mu L} \right) \left(\frac{3}{R_0^4} \cdot \frac{R_0 - R_L}{R_L^{-3} - R_0^{-3}} \right)$$

This is the result, with the first factor being the solution for a straight tube, the second factor being a correction factor. It would be better to write the correction factor as "1 - X", so that the quantity X gives the deviation from straight-tube behavior. The quantity X is then

$$\begin{aligned}
X &= 1 - \frac{3}{R_0^4} \cdot \frac{R_0 - R_L}{R_L^{-3} - R_0^{-3}} = 1 - \frac{3[1 - (R_L/R_0)]}{(R_0/R_L)^3 - 1} = 1 - \frac{3[1 - (R_L/R_0)](R_L/R_0)^3}{1 - (R_L/R_0)^3} \\
&= 1 - \frac{3(R_L/R_0)^3}{1 + (R_L/R_0) + (R_L/R_0)^2} = \frac{1 + (R_L/R_0) + (R_L/R_0)^2 - 3(R_L/R_0)^3}{1 + (R_L/R_0) + (R_L/R_0)^2}
\end{aligned}$$

which then leads to the desired result in Eq. 2B.10-3.

2B.11 The cone-and-plate viscometer

a. In a parallel-plate system with rectilinear flow, the velocity distribution is just $v_x/v_0 = y/b$, where b is the plate spacing and v_0 is the velocity of the upper plate. We now make the following correspondences between the parallel-plate system and the cone-plate system (using θ as the usual variable in spherical coordinates measured downward from the z -axis, and ψ as the variable measured upward from the plate surface):

$$v_x \leftrightarrow v_\phi; \quad v_0 \leftrightarrow \Omega r; \quad b \leftrightarrow r \sin \psi_0; \quad y \leftrightarrow r \sin \psi \approx r\psi = r\left(\frac{1}{2}\pi - \theta\right)$$

When this correspondence is made, Eq. 2B.11-1 results.

b. From Eq. B.1-19, we get for the force per unit area in the ϕ -direction on a face perpendicular to the θ -direction

$$\tau_{\theta\phi} = -\mu \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) \approx -\mu \frac{1}{r} \frac{\partial v_\phi}{\partial \theta} = -\mu \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{\Omega r \left(\frac{1}{2}\pi - \theta\right)}{\psi_0} \right)$$

Here we have used the fact that the angle between the cone and plate is so tiny that θ is very nearly $\frac{1}{2}\pi$ so that $\sin \theta$ is very close to unity.

c. The torque is obtained by integrating the force times lever arm over the entire plate area:

$$T_z = \int_0^{2\pi} \int_0^R \left(\tau_{\theta\phi} \cdot r \right) \Big|_{\theta=\pi/2} r dr d\phi = 2\pi \left(\frac{\mu\Omega}{\psi_0} \right) \int_0^R r^2 dr = 2\pi \left(\frac{\mu\Omega}{\psi_0} \right) \left(\frac{R^3}{3} \right)$$

which leads to Eq. 2B.11-3.

2B.12 Flow of a fluid in a network of tubes

At A the pipe splits into three pipes, and at the next set of junctions the fluid flows equally in six pipes, and then at the next set of junctions the fluid flows back into three pipes, and finally at B the fluid is all returned to a single pipe. Call the modified pressure at the junctions where three pipes split into six pipes $\mathcal{P}_{3 \rightarrow 6}$, and that where six pipes join to form three pipes $\mathcal{P}_{6 \rightarrow 3}$.

Then in each of the first set of three pipes

$$\frac{w}{3} = \frac{\pi(\mathcal{P}_A - \mathcal{P}_{3 \rightarrow 6})R^4\rho}{8\mu L} \quad \text{or} \quad \mathcal{P}_A - \mathcal{P}_{3 \rightarrow 6} = \frac{8\mu L w}{3\pi R^4\rho}$$

In each of the batch of six pipes

$$\frac{w}{6} = \frac{\pi(\mathcal{P}_{3 \rightarrow 6} - \mathcal{P}_{6 \rightarrow 3})R^4\rho}{8\mu L} \quad \text{or} \quad \mathcal{P}_{3 \rightarrow 6} - \mathcal{P}_{6 \rightarrow 3} = \frac{8\mu L w}{6\pi R^4\rho}$$

and in each of the final batch of 3 pipes

$$\frac{w}{3} = \frac{\pi(\mathcal{P}_{6 \rightarrow 3} - \mathcal{P}_B)R^4\rho}{8\mu L} \quad \text{or} \quad \mathcal{P}_{6 \rightarrow 3} - \mathcal{P}_B = \frac{8\mu L w}{3\pi R^4\rho}$$

When all the pressure differences are added together, the unknown quantities $\mathcal{P}_{3 \rightarrow 6}$ and $\mathcal{P}_{6 \rightarrow 3}$ cancel out, and we get

$$\mathcal{P}_A - \mathcal{P}_B = \frac{8\mu L w}{\pi R^4\rho} \left(\frac{5}{6} \right) \quad \text{or} \quad w = \frac{3\pi(\mathcal{P}_A - \mathcal{P}_B)R^4\rho}{20\mu L}$$

2C.1 Performance of an electric dust collector

a. First we solve the problem of the vertical motion of the particle as it falls under the action of the electromagnetic field. The equation of motion for the particle (without gravitational acceleration or Stokes drag) is

$$e\mathcal{E} = m \frac{d^2x}{dt^2}$$

This equation may be integrated with the initial conditions that $x = x_0$ and $dx/dt = 0$ at $t = 0$, to give

$$x = x_0 - \frac{e\mathcal{E}t^2}{2m}$$

From this we can get the time t_f required for the particle to fall to $x = -B$:

$$t_f = \sqrt{\frac{2m(B + x_0)}{e\mathcal{E}}}$$

Next we look at the horizontal motion. From Eq. 2B.3-2 and the expression for $x(t)$, we find that (with $v_m = (p_0 - p_L)B^2/2\mu L$)

$$\frac{dz}{dt} = v_m \left[1 - \left(\frac{x}{B} \right)^2 \right] = \frac{dz}{dt} = v_m \left[1 - \left(\frac{x_0 - (e\mathcal{E}t^2/2m)}{B} \right)^2 \right]$$

This may be integrated to give

$$\begin{aligned} L = \int_0^L dz &= v_m \int_0^{t_f} \left[1 - \frac{1}{B^2} \left(x_0^2 - \frac{x_0 e\mathcal{E}t^2}{m} + \frac{e^2 \mathcal{E}^2 t^4}{4m^2} \right) dt \right] \\ &= \frac{(p_0 - p_L)}{2\mu L} \left[(B^2 - x_0^2)t_f + \frac{x_0 e\mathcal{E}t_f^3}{3m} - \frac{e^2 \mathcal{E}^2 t_f^5}{20m^2} \right] \end{aligned}$$

Next, square this expression and then insert the expression for t_f above to get

$$L^2 = \frac{2(p_0 - p_L)}{15\mu} \sqrt{\frac{2m(B + x_0)}{e\mathcal{E}}} (3B - 2x_0)(B + x_0)$$

Then, in order to remove the radical, we square this, thereby getting

$$L^4 = \frac{8(p_0 - p_L)^2 m}{225\mu^2 e\mathcal{E}} (3B - 2x_0)^2 (B + x_0)^3$$

Next, we set dL^4/dx_0 equal to zero, and this yields 4 values for x_0 : $\frac{1}{2}B$, $\frac{3}{2}B$, B , and B . It is only the first of these that is physically acceptable. When that value is put back into the expression for L , we get finally

$$L_{\min} = \left[\frac{12 (p_0 - p_L)^2 m B^5}{35 \mu^2 e\mathcal{E}} \right]^{1/4}$$

2C.2 Residence-time distribution in tube flow

a. A fluid element at a radius r within the tube will require a time t to reach the tube outlet

$$t(r) = \frac{L}{v_z(r)} = \frac{L}{v_{\max} \left[1 - (r/R)^2 \right]}$$

All the fluid with a radius less than r will have left the tube at this time. Hence the fraction of the flow that will have left the tube is

$$F(t) = \frac{\int_0^{2\pi} \int_0^r v_z r dr d\theta}{\int_0^{2\pi} \int_0^R v_z r dr d\theta} = 2 \left(\frac{r}{R} \right)^2 - \left(\frac{r}{R} \right)^4$$

When the first equation is solved for $(r/R)^2$ and substituted into the expression for $F(t)$, we get

$$F(t) = 2 \left(1 - \frac{L}{v_{\max} t} \right) - \left(1 - \frac{L}{v_{\max} t} \right)^2 = 1 - \left(\frac{L}{v_{\max} t} \right)^2 = 1 - \left(\frac{L}{2 \langle v_z \rangle t} \right)^2$$

b, The mean residence time is then obtained by solving the last equation in (a) for t and substituting into the Eq. 2C.2-1:

$$t_m = \int_0^1 t dF = \frac{L}{v_{\max}} \int_0^1 \frac{dF}{\sqrt{1-F}} = \frac{2L}{v_{\max}} = \frac{L}{\langle v_z \rangle}$$

2C.3 Velocity distribution in a tube

The derivation in §2.3 is valid up through Eq. 2.3-15. If the viscosity is dependent on the radial coordinate, however, Eq. 2.3-16 is inappropriate. Instead we get

$$v_z = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{2L} \int_0^r \frac{\bar{r}}{\mu(\bar{r})} d\bar{r} + C_2$$

Application of the no-slip boundary condition at the tube wall gives

$$0 = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{2L} \int_0^R \frac{\bar{r}}{\mu(\bar{r})} d\bar{r} + C_2$$

This may be solved for the integration constant, and the velocity distribution is then

$$v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{2L} \int_r^R \frac{\bar{r}}{\mu(\bar{r})} d\bar{r} \quad \text{or} \quad v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{2L} \int_y^1 \frac{\bar{y}}{\mu(\bar{y})} d\bar{y}$$

This is the same as Eq. 2.3-18 if the viscosity is a constant.

Next we get an expression for the average velocity

$$\begin{aligned} \langle v_z \rangle &= \frac{\int_0^{2\pi} \int_0^R v_z r dr d\theta}{\int_0^{2\pi} \int_0^R r dr d\theta} = \frac{2}{R^2} \int_0^R v_z r dr = 2 \int_0^1 v_z y dy \\ &= \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{L} \int_0^1 \int_y^1 \frac{\bar{y}}{\mu(\bar{y})} d\bar{y} y dy = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{L} \int_0^1 \frac{\bar{y}}{\mu(\bar{y})} \int_0^{\bar{y}} y dy d\bar{y} \\ &= \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{L} \int_0^1 \frac{\bar{y}}{\mu(\bar{y})} \frac{\bar{y}^2}{2} d\bar{y} \end{aligned}$$

Then we find the dimensionless ratio $\frac{v_z}{\langle v_z \rangle} = \frac{\int_y^1 (\bar{y}/\mu) d\bar{y}}{\int_0^1 (\bar{y}^3/\mu) d\bar{y}}$.

2C.4 Falling-cylinder viscometer

a. Equation 2.4-2 is valid for this problem, but the pressure difference is not known. When Newton's law of viscosity is substituted into Eq. 2.4-2 we get

$$v_z = -\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4\mu L}\right)r^2 - C_1 \ln r + C_2$$

The two constants of integration and the (unknown) pressure difference can be obtained from two boundary conditions and a mass-conservation condition: At $r = \kappa R$, $v_z = -v_0$; at $r = R$, $v_z = 0$; and $\int_0^{2\pi} \int_{\kappa R}^R v_z r dr d\theta = \pi(\kappa R)^2 v_0$. This states that the fluid displaced by the falling cylinder must be compensated for by a net motion upward through the annular slit. These three unknown constants may be obtained from these conditions (lengthy!) and the result is

$$\frac{v_z}{v_0} = -\frac{(1 - \xi^2) - (1 + \kappa^2)\ln(1/\xi)}{(1 - \kappa^2) - (1 + \kappa^2)\ln(1/\kappa)}$$

b. The force acting downward on the cylindrical slug of height H is $(\rho_0 - \rho)g \cdot \pi(\kappa R)^2 H$. The difference in the pressures acting on the top and bottom of the slug is an upward force

$$\begin{aligned} (\mathcal{P}_0 - \mathcal{P}_H) \cdot \pi(\kappa R)^2 &= -\int_0^H (d\mathcal{P}/dz) dz \cdot \pi(\kappa R)^2 \\ &= -\frac{4v_0 H \cdot \pi(\kappa R)^2}{R^2 \left[(1 - \kappa^2) - (1 + \kappa^2)\ln(1/\kappa) \right]} \end{aligned}$$

In addition, there is an upward force associated with the frictional drag by the fluid

$$\begin{aligned} 2\pi(\kappa R)H(-\tau_{rz})\Big|_{r=\kappa R} &= 2\pi(\kappa R)H\mu \left(\frac{dv_z}{dr} \right) \Big|_{r=\kappa R} \\ &= 2\pi\kappa v_0 H\mu \left(\frac{2\kappa - (1 + \kappa^2)(1/\kappa)}{(1 - \kappa^2) - (1 + \kappa^2)\ln(1/\kappa)} \right) \end{aligned}$$

When these are equated and the result solved for the viscosity, the expression in Eq. 22C.4-2 is obtained.

c. Next put $\kappa = 1 - \varepsilon$, and expand in powers of ε , keeping terms up to ε^5 . Use §C.2, and obtain Eq. 2C.4-3.

2C.5 Falling film on a conical surface

a. A mass balance on a ring of liquid contained between s and $s + \Delta s$ gives

$$(2\pi s(\sin\beta)\delta(s)\langle v \rangle)\Big|_s - (2\pi s(\sin\beta)\delta(s)\langle v \rangle)\Big|_{s+\Delta s} = 0$$

Letting $\Delta s \rightarrow 0$ then gives

$$\frac{d}{ds}(s\delta\langle v \rangle) = 0 \quad \text{whence, from Eq. 2.2-20} \quad \frac{d}{ds}(s\delta^3) = 0$$

Equation 2.2-20 is valid strictly for a flat plate with constant film thickness, but we apply it here *approximately* to a different geometry.

b. When the equation in (a) is integrated, we get $s\delta^3 = C$, in which C is an integration constant. This constant is determined by requiring that the mass flow down the conical surface be the same as that flowing up the central tube (i.e., w). We hence write (width of film) \times (thickness of film) \times (mass flow rate), and then use Eq. 2.2-20:

$$w = (2\pi s \sin\beta) \cdot \delta \cdot \rho \langle v \rangle = (2\pi s \sin\beta) \cdot \left(\frac{C}{s}\right)^{1/3} \cdot \rho \left(\frac{\rho g (C/s)^{2/3} \cos\beta}{3\mu}\right)$$

From this we get C ;

$$C = \frac{3\mu w}{(2\pi \sin\beta) \cdot (\rho^2 g \cos\beta)} = \frac{3\mu w}{\pi \rho^2 g \sin 2\beta}$$

The film thickness as a function of the distance down the cone from the apex is thus

$$\delta = \sqrt[3]{\frac{3\mu w}{\pi \rho^2 g s \sin 2\beta}}$$

2C.6 Rotating cone pump

a. Inner cone not rotating.

For sufficiently small values of B , the flow will resemble very much that for a thin slit (see Problem 2B.3), for which the mass rate of flow is given as the answer to part (b). This formula may be adapted to the flow in the annular space of height dz . as follows, if the inner cone is not rotating and if the gravitational force is not included:

$$w \approx \frac{2}{3} \left(-\frac{dp}{dz} \right) \frac{B^3 (2\pi z \sin \theta) \rho}{L}$$

where we have made the identification $(p_0 - p_L)/L \rightarrow -dp/dz$ and $W \rightarrow 2\pi r = 2\pi z \sin \theta$. Across any plane $z = \text{constant}$, the mass flow rate will be a constant. Hence the above equation can be integrated to give

$$-\int_{p_1}^{p_2} dp = \frac{3}{4\pi} \frac{\mu w}{B^3 \rho \sin \theta} \int_{L_1}^{L_2} \frac{dz}{z} \quad \text{or} \quad p_1 - p_2 = \frac{3}{4\pi} \frac{\mu w}{B^3 \rho \sin \theta} \ln \frac{L_2}{L_1}$$

b. Effect of the gravitational force and the centrifugal force

The result in (a) may be modified to include the effect of gravitational acceleration g and the angular velocity Ω of the inner cone.

The gravity force in the z -direction (per unit volume) is given by $F_{\text{grav},z} = -\rho g \cos \beta$. The centrifugal force (per unit volume) acting in the middle of the slit will be, approximately, $F_{\text{cent}} = \rho (\frac{1}{2} \Omega r)^2 / r = \frac{1}{4} \rho \Omega^2 z \sin \beta$, where r is the distance from the centerline of the cones to the middle of the slit. The component of this force in the z -direction is then $F_{\text{cent},z} = \frac{1}{4} \rho \Omega^2 z \sin^2 \beta$. Then the first equation in (a) can be modified to give

$$w \approx \frac{2}{3} \left(-\frac{dp}{dz} + \frac{1}{4} \rho \Omega^2 z \sin^2 \beta - \rho g \cos \beta \right) \frac{B^3 (2\pi z \sin \beta) \rho}{L}$$

This equation can be integrated to give

$$w = \frac{4\pi B^3 \rho \sin \beta}{3\mu \ln(L_2/L_1)} \left[(p_1 - p_2) + \left(\frac{1}{8}\rho\Omega^2 \sin^2 \beta\right)(L_2^2 - L_1^2) - (\rho g \cos \beta)(L_2 - L_1) \right]$$

Many assumptions have been used to get this solution: (1) laminar flow (turbulent flow analog is not difficult to work out); (2) curvature effects have been neglected (correction for this is easy to do); (3) entrance effects have been ignored (this can probably be handled approximately by introducing an "equivalent length"); (4) instantaneous accommodation of velocity profiles to the changing cross-section (it would be difficult to correct for this in a simple way).

2C.7 A simple rate-of-climb indicator

a. Consider two planes of area S parallel to the earth's surface at heights z and $z + \Delta z$. The pressure force in the z -direction acting on the plane at height z and that acting at the plane at height $z + \Delta z$ will be just the mass of air in the layer of height Δz :

$$Sp|_z - Sp|_{z+\Delta z} = \rho g S \Delta z$$

Division by $S\Delta z$ and then letting $\Delta z \rightarrow 0$ gives the differential equation

$$\frac{dp}{dz} = -\rho g \quad \text{or} \quad \frac{dp}{dz} = -\left(\frac{pM}{R_g T}\right)g$$

which describes the decrease in the atmospheric pressure with increased elevation; here R_g is the gas constant.

b. Let p_i be the pressure inside the Bourdon element and p_o be the pressure outside (i.e., the ambient atmospheric pressure). We now write an equation of conservation of mass for the entire instrument:

$$\frac{d}{dt}m_{\text{tot}} = -w_o; \quad V \frac{d}{dt}\rho_i = \frac{\pi(p_i - p_o)R^4}{8\mu L} \rho_{\text{avg}}; \quad V \frac{d}{dt}p_i = \frac{\pi(p_i - p_o)R^4}{8\mu L} p_{\text{avg}}$$

Here m_{tot} is the total mass of air within the system (Bourdon element plus capillary tube), w_o is the mass rate of flow of the air exiting to the outside, ρ_i is the density of the air inside the Bourdon elements, and ρ_{avg} is the arithmetic average of the inlet and outlet densities within the capillary tube (see Eq. 2.3-29). The third form of the mass balance written above has made use of the ideal gas law, $p = \rho R_g T / M$.

If we neglect changes in the arithmetic average pressure p_{avg} and use the abbreviation $B = \pi R^4 p_{\text{avg}} / 8\mu L V$, we can integrate the mass balance above and get

$$p_i = e^{-Bt} \left(\int B p_o e^{Bt} dt + C \right)$$

To get $p_o(t)$ we make use of the fact that there is a constant upward velocity, so that

$$\frac{dp_o}{dt} = \frac{dp_o}{dz} \frac{dz}{dt} = -\left(\frac{p_o M}{R_g T}\right) g \cdot v_z \equiv -A p_o \quad \text{whence} \quad p_o = p_o^0 e^{-At}$$

Then the mass-balance equation becomes

$$p_i = e^{-Bt} \left(\int B p_o^0 e^{-At} e^{Bt} dt + C \right) = \frac{B p_o^0}{B-A} e^{-At} + C e^{-Bt}$$

Determine the constant of integration, C , from the initial condition that $p_i^0 = p_o^0$ at $t = 0$. Then

$$p_i = p_o^0 \frac{B e^{-At} - A e^{-Bt}}{B-A} \quad \text{and} \quad P = \frac{p_i - p_o}{p_o} = \frac{A}{B-A} \left(1 - e^{-(B-A)t} \right)$$

In the limit that $t \rightarrow \infty$, we get for $B \gg A$

$$P \rightarrow \frac{A}{B} = \frac{v_z M g}{R_g T} \cdot \frac{8V\mu L}{\pi R^4 p_{\text{avg}}}$$

Hence for $p_o \approx p_{\text{avg}}$, the gauge pressure is

$$p_i - p_o = v_z \left(\frac{8\mu L}{\pi R^4} \right) \left(\frac{MgV}{R_g T} \right)$$

Hence the pressure difference approaches an asymptotic value that varies only slightly with altitude.

c. To get the relaxation time, note that

$$P = \frac{A}{B-A} \left(1 - e^{-(B-A)t} \right) \xrightarrow{B \gg A} P_\infty \left(1 - e^{-Bt} \right)$$

whence

$$\frac{P_{\infty} - P}{P_{\infty}} = e^{-Bt} \equiv e^{-t/t_{\text{rel}}} \quad \text{so that} \quad t_{\text{rel}} = \frac{1}{B} = \frac{8\mu LV}{\pi R^4 p_{\text{avg}}}$$

It is necessary to have $t_{\text{rel}} \ll 100$ to insure the plausibility of the assumption that $B \gg A$.

2D.1 Rolling-ball viscometer

The rolling-ball viscometer consists of an inclined tube containing a sphere whose diameter is but slightly smaller than the internal diameter of the tube. The fluid viscosity is determined by observing the speed with which the ball rolls down the tube, when the latter is filled with liquid. We want to interrelate the viscosity and the terminal velocity of the rolling ball.

The flow between the sphere and the cylinder can be treated locally as slit flow (see Problem 2B.3) and hence the only hydrodynamic result we need is

$$\frac{dp}{dz} = \frac{12\mu\langle v_z \rangle}{\sigma^2} \quad (*)$$

But we must allow the slit width σ to vary with θ and z . From the figure we see that

$$R^2 = (R-r)^2 + (r' + \sigma)^2 - 2(R-r)(r' + \sigma)\cos\theta$$

where $r' = \sqrt{r^2 - z^2}$. Solving for σ we get

$$\sigma = -r' + (R-r)\cos\theta + R\sqrt{1 + [(R-r)/R]^2(-\sin^2\theta)}$$

The second term under the square-root sign will be very small for the tightly fitting sphere-cylinder system and will hence be neglected. Furthermore we replace $\sqrt{r^2 - z^2}$ by $\sqrt{R^2 - z^2}$ and add compensating terms

$$\begin{aligned} \sigma &= (R-r) \left[\cos\theta + \frac{R - \sqrt{R^2 - z^2} + (\sqrt{R^2 - z^2} - \sqrt{r^2 - z^2})}{R-r} \right] \\ &= (R-r) \left[\cos\theta + \frac{R - \sqrt{R^2 - z^2} + (R-r) - \frac{1}{2}(R-r)(z/R)^2 \dots}{R-r} \right] \\ &\approx (R-r) \left[(\cos\theta + 1) + \frac{R - \sqrt{R^2 - z^2}}{R-r} \right] \end{aligned}$$

$$= 2(R-r) \left(\cos^2 \frac{\theta}{2} + \frac{R - \sqrt{R^2 - z^2}}{2(R-r)} \right)$$

The omission of the term containing $(z/R)^2$ and the higher-order terms is possible, since the greatest contribution to the viscous drag occurs at the plane $z = 0$, and hence less accuracy is required for regions of larger z . Note that the above result gives correctly $\sigma = 0$ at $z = 0$, $\theta = \pi$, and $\sigma = 2(R-r)$ at $z = 0$, $\theta = 0$.

Next we assert that dp/dz will be independent of θ , which is probably a good approximation. Then according to (*) $\langle v_z \rangle$ must have the form

$$\langle v_z \rangle = B(z) \sigma^2 \quad (**)$$

Next, the volume rate of flow across any plane z will be

$$\begin{aligned} Q &= \int_{-\pi}^{+\pi} \langle v_z \rangle \sigma(\theta, z) R d\theta = RB(z) \int_{-\pi}^{+\pi} [\sigma(\theta, z)]^3 d\theta \\ &= 8RB(z)(R-r)^3 \int_{-\pi}^{+\pi} \left[\cos^2 \frac{1}{2} \theta + \alpha \right]^3 d\theta = 8RB(z)(R-r)^3 I(\alpha) \end{aligned}$$

in which $\alpha = (R - \sqrt{R^2 - z^2}) / 2(R-r)$.

The volume rate of flow Q at all cross-sections will be the same, and its value will be, to a very good approximation $Q = \pi R^2 v_0$, where v_0 is the translational speed of the rolling ball. Equating the two expressions for Q gives

$$B(z) = \frac{\pi R v_0}{4z(R-r)^3 I(\alpha)} \quad (***)$$

Combining (*), (**), and (***) we get

$$\frac{dp}{dz} = \frac{3\pi\mu R v_0}{2(R-r)^3 I(\alpha)}$$

The total pressure drop across the slit is then

$$\Delta p = \int_{-R}^{+R} \frac{dp}{dz} dz \approx 2 \int_0^{\infty} \frac{dp}{dz} \frac{dz}{d\alpha} d\alpha$$

into which we have to insert $dz/d\alpha$. Virtually no error is introduced by making the upper limit infinite. From the definition of α

$$z^2 = -4(R-r)^2 \alpha^2 + 4R(R-r)\alpha$$

The first term on the right is smaller than the second, at least for small z . Then $dz \approx \sqrt{R(R-r)} d\alpha / \sqrt{\alpha}$, and the pressure drop expression becomes (with $\xi^2 = \alpha$)

$$\begin{aligned} \Delta p &= 2\sqrt{R(R-r)} \int_0^{\infty} \frac{dp}{dz} \frac{1}{\sqrt{\alpha}} d\alpha = 4\sqrt{R(R-r)} \int_0^{\infty} \frac{dp}{dz} d\xi \\ &= 4\sqrt{R(R-r)} \frac{3\pi\mu v_0 R}{2(R-r)^3} \int_0^{\infty} \frac{1}{I(\xi^2)} d\xi = \frac{3\pi\mu v_0 R^{3/2}}{2(R-r)^{5/2}} \cdot J \end{aligned}$$

where

$$J = 2 \int_0^{\infty} \frac{1}{I(\xi^2)} d\xi = \frac{4}{3} \left[\sqrt{2} - \frac{1}{\sqrt{5}} (\sqrt{10} + 2)^{1/2} \right] = 0.531$$

The pressure drop multiplied by the tube cross-section must, according to an overall force balance, be equal to the net force acting on the sphere by gravity and buoyancy

$$\frac{4}{3} \pi R^3 (\rho_s - \rho) g \sin \beta = \pi R^2 \Delta p$$

where ρ_s and ρ are the densities of the sphere and fluid respectively. Combining the last three results gives the equation for the viscosity

$$\mu = \frac{4}{9\pi J} \frac{R^2 (\rho_s - \rho) g \sin \beta}{v_0} \left(\frac{R-r}{R} \right)^{5/2}$$

2D.2 Drainage of liquids

a. The unsteady mass balance is

$$\frac{\partial}{\partial t}(\rho\delta W) = (\rho\langle v_z \rangle W\delta)\Big|_z - (\rho\langle v_z \rangle W\delta)\Big|_{z+\Delta z}$$

Divide by $\rho W \Delta z$ and take the limit as $\Delta z \rightarrow 0$, to get Eq. 2D.2-1.

b. Then use Eq. 2.2-22 to get Eq. 2D.2-1:

$$\frac{\partial \delta}{\partial t} = -\frac{\rho g}{3\mu} \frac{\partial \delta^3}{\partial z} = -\frac{\rho g \delta^2}{\mu} \frac{\partial \delta}{\partial z}$$

which is a first-order partial differential equation.

c. First let $\Delta = \sqrt{\rho g / \mu}$, so that the equation in (b) becomes:

$$\frac{\partial \Delta}{\partial t} = \Delta^2 \frac{\partial \Delta}{\partial z}$$

Inspection of the equation suggests that $\Delta = \sqrt{z/t}$, which can be seen to satisfy the differential equation exactly. Therefore Eq. 2D.2-3 follows at once. This equation has a reasonable form, since for long times the boundary layer is thin, whereas for short times the boundary layer is thick.

3A.1 Torque required to turn a friction bearing.

Equation 3.6-31 describes the torque required to turn an *outer* cylinder at an angular velocity Ω_o . The corresponding expression for the torque required to turn an *inner* rotating cylinder at an angular velocity Ω_i is given by a formally similar expression,

$$T_z = 4\pi\mu\Omega_i R^2 L \left(\frac{\kappa^2}{1 - \kappa^2} \right)$$

derivable in like manner from the corresponding velocity profile in Eq. 3.6-32.

The specifications for this problem (converted into SI units via Appendix F) are:

$$\begin{aligned} \kappa &= \frac{1.000}{1.002} = 0.998004; & \kappa^2 &= 0.996012 \\ \left(\frac{\kappa^2}{1 - \kappa^2} \right) &= \frac{0.996012}{0.003988} = 249.8 \\ \mu &= (200 \text{ cp})(10^{-3} \text{ kg/m}\cdot\text{s}/\text{cp}) = 0.200 \text{ kg/m}\cdot\text{s} \\ \Omega_i &= (200 \text{ rpm})(1 \text{ min}/60 \text{ s})(2\pi \text{ radians/revolution}) = 20\pi/3 \text{ radians/s} \\ R^2 &= (1 \text{ in}^2)(1 \text{ m}/39.37 \text{ in})^2 = 0.000645 \text{ m}^2 \\ L &= 2 \text{ in} = 2/39.37 \text{ m} = 0.0508 \text{ m} \\ \rho &= (50 \text{ lb}_m/\text{ft}^3)(0.45359 \text{ kg}/\text{lb}_m(39.370/12 \text{ ft}/\text{m})^3) = 800.9 \text{ kg/m}^3 \end{aligned}$$

Hence, the required torque is

$$\begin{aligned} T_z &= (4\pi)(0.200 \text{ kg/m}\cdot\text{s})(20\pi/3 \text{ radians/s})(0.000645 \text{ m}^2)(0.0508 \text{ m})(249.8) \\ &= 0.431 \text{ kg}\cdot\text{m}^2/\text{s}^2 = 0.32 \text{ ft}\cdot\text{lb}_f \end{aligned}$$

and the power required is

$$\begin{aligned} P &= T_z \Omega_i = (0.32 \text{ lb}_f\cdot\text{ft})(20\pi/3 \text{ s}^{-1})(3600 \text{ s/hr})(5.0505 \times 10^{-7} \text{ hp}\cdot\text{hr}/\text{lb}_f\cdot\text{ft}) \\ &= 0.012 \text{ hp} \end{aligned}$$

In these calculations we have tacitly assumed the flow to be stable and laminar. To test this assumption, we formulate a transition criterion based on the critical angular velocity expression given under Fig. 3.6-2:

$$\text{Re} := \frac{\Omega_i \rho R^2 (1 - \kappa)^{3/2}}{\mu} < \text{about } 41.3 \text{ for } \kappa \approx 1.$$

Insertion of numerical values for the present system gives

$$\text{Re} = \frac{(20\pi/3 \text{ radians/s})(800.9 \text{ kg/m}^3)(0.000645 \text{ m}^2)(1 - 0.998004)^{3/2}}{(0.200 \text{ kg/m}\cdot\text{s})} = 0.0048$$

This Re value is well below the transition value of 41.3 for this geometry; therefore, the foregoing predictions of T_z and P are realistic.

3A.2 Friction loss in bearings.

The power expended to overcome the bearing friction is

$$P = T_z \Omega_i = 4\pi\mu\Omega_i^2 R^2 L \left(\frac{\kappa^2}{1 - \kappa^2} \right)$$

in which L is the total bearing length of $2 \times 20 \times 1 = 40$ ft for the two shafts. The specifications for this problem (converted to SI units via Appendix F) are:

$$\begin{aligned}\kappa &= \frac{16}{16 + 2 \times 0.005} = 0.999375; & \kappa^2 &= 0.998751 \\ \left(\frac{\kappa^2}{1 - \kappa^2} \right) &= \frac{0.998751}{0.001249} = 799.6 \\ \mu &= (5000 \text{ cp})(10^{-3} \text{ kg/m}\cdot\text{s}/\text{cp}) = 5 \text{ kg/m}\cdot\text{s} \\ \Omega_i &= (50/60 \text{ rev/s})(2\pi \text{ radians/rev}) = 5\pi/3 \text{ radians/s} \\ R^2 &= (8/39.37 \text{ m})^2 = 0.04129 \text{ m}^2 \\ L &= 40 \text{ ft} = (40 \times 12/39.37 \text{ m}) = 12.2 \text{ m}\end{aligned}$$

With these values, the calculated power requirement is

$$\begin{aligned}P &= (4\pi)(5 \text{ kg/m}\cdot\text{s})(5\pi/3 \text{ rad/s})^2(0.04129 \times 12.2 \text{ m}^3)(799.6) \\ &= 6.938 \times 10^5 \text{ kg}\cdot\text{m}^2/\text{s}^3\end{aligned}$$

This result is then expressed in horsepower by use of Table F.3-3:

$$\begin{aligned}P &= (6.938 \times 10^5 \text{ kg}\cdot\text{m}^2/\text{s}^3)(3.7251 \times 10^{-7} \text{ hp}\cdot\text{hr}\cdot[\text{kg}\cdot\text{m}^2/\text{s}^2]^{-1})(3600 \text{ s/hr}) \\ &= 930 \text{ hp}\end{aligned}$$

Thus, the fraction of the available power that is lost in bearing friction is $930/(4000 + 4000) = 0.116$.

3A.3 Effect of altitude on air pressure.

For a stationary atmosphere (i.e., no wind currents), the vertical component of the equation of motion gives

$$\frac{dp}{dz} = -\rho g$$

The air is treated as an ideal gas,

$$\rho = \frac{pM}{RT}$$

with $M \approx 29$, and with temperature in °R given by

$$T(z) = 530 - 0.003z$$

at elevation z ft above Lake Superior. The pressure p_2 at $z_2 = 2023 - 602 = 1421$ ft above lake level is to be calculated, given that $p_1 = 750$ mm Hg at $z = 0$.

The foregoing equations give

$$\frac{d \ln p}{dz} = -\frac{Mg}{R(530 - 0.003z)}$$

Integration gives

$$\begin{aligned} \ln(p_2/p_1) &= \frac{Mg}{R} \int_0^{1421} \frac{dz}{(530 - 0.003z)} \\ &= \frac{Mg}{R} \frac{1}{0.003} \ln[530 - 0.003z] \Bigg|_{z=0}^{z=1421} \\ &= \frac{Mg}{0.003R} \ln \left[\frac{525.737}{530} \right] \end{aligned}$$

Insertion of numerical values in lb_m-ft-s units gives

$$\begin{aligned} \ln(p_2/p_1) &= \frac{(29 \text{ lb}_m/\text{lb-mol})(32.17 \text{ ft/s}^2)}{(0.003 \text{ R/ft})(4.9686 \times 10^4 \text{ lb}_m \cdot \text{ft}^2/\text{s}^2 \cdot \text{lb-mol} \cdot \text{R})} \ln [525.737/\text{over}530] \\ &= -0.0505 \end{aligned}$$

Hence,

$$p_2 = p_1 \exp(-0.0505) = 750 \times 0.9507 = 713 \text{ mm Hg}$$

Since the fractional change in P is small, one gets a good approximation (and a quicker solution) by neglecting it. That method gives $p_2 = 712$ mm Hg.

3A.4 Viscosity determination with a rotating-cylinder viscometer.

Here it is desirable to use a sufficiently high torque that the precision of viscosity determinations is limited mainly by that of the measurement of angular velocity. A torque of 10^4 dyn-cm, corresponding to a torque uncertainty of 1%, appears reasonable if the resulting Reynolds number is in the stable laminar range.

The geometric specifications of the viscometer are:

$$\begin{aligned}R &= 2.25 \text{ cm}; & \kappa R &= 2.00 \text{ cm} \\ \kappa &= 2.00/2.25 = 0.888889; & \kappa^2 &= 0.790123 \\ 1 - \kappa^2 &= 0.209877; & (\kappa R)^2 &= 4.00 \text{ cm}^2 \\ L &= 4 \text{ cm}; & R^2 &= 5.0625 \text{ cm}^2\end{aligned}$$

The angular velocity corresponding to this torque value is:

$$\Omega_o = \frac{T_z(1 - \kappa^2)}{4\pi\mu(\kappa R)^2 L} = \frac{(10^4 \text{ g}\cdot\text{cm}^2/\text{s}^2)(0.209877)}{4\pi(0.57 \text{ g}/\text{cm}\cdot\text{s})(4 \text{ cm}^2)(4 \text{ cm})} = 18.3 \text{ radians/s}$$

The Reynolds number at this condition is:

$$\text{Re} = \frac{\Omega_o R^2 \rho}{\mu} = \frac{(18.3)(5.0625)(1.29)}{0.57} = 210$$

According to Fig. 3.6-2, this Re value is well within the stable laminar range; therefore, a torque of 10^4 dyn-cm is acceptable.

3A.5 Fabrication of a parabolic mirror.

Equation 3.6-44 gives the shape of the free surface as

$$z - z_0 = \left(\frac{\Omega^2}{2g} \right) r^2$$

The required derivatives of this function at the axis of rotation are

$$\frac{dz}{dr} = 0 \quad \text{and} \quad \frac{d^2z}{dr^2} = \frac{\Omega^2}{g}$$

Setting the desired focal length equal to half the radius of curvature of the mirror surface at $r = 0$, and using Eq. 3A.5-1, we obtain

$$f = \frac{1}{2}g/\Omega^2$$

Thus, the required angular velocity to produce a mirror with focal length $f = 100$ cm at standard terrestrial gravity is

$$\begin{aligned} \Omega &= \sqrt{\frac{g}{2f}} \\ &= \sqrt{\frac{980.665 \text{ cm/s}^2}{(2)(100 \text{ cm})}} \\ &= 2.214 \text{ radians/s} \end{aligned}$$

which corresponds to $60\Omega/2\pi = 21.1$ revolutions per minute.

3A.6 Scale-up of an agitated tank.

The specifications for the operation in the large tank (Tank I) are

$$N_I = 120 \text{ rpm}; \quad \mu_I = 13.5 \text{ cp}; \quad \rho_I = 0.9 \text{ g/cm}^3$$

and the tank is to be operated with an uncovered liquid surface.

To allow direct prediction of the operation of Tank I from experiments in the smaller system (Tank II), the systems must be geometrically similar and must run at the same values of Re and Fr. To meet the latter requirement, Eqs. 3.7-40,41 must be satisfied.

Equation 3.7-41 requires

$$D_{II}N_{II}^2 = D_I N_I^2$$

when, as usual, the gravitational fields for the two systems are equal. Then the model must operate at

$$N_{II} = N_I \sqrt{D_I/D_{II}} = 120\sqrt{10} = 380 \text{ rpm}$$

and Eq. 3.7-40 requires

$$\begin{aligned} \nu_{II} &= \nu_I \left(\frac{D_{II}}{D_I} \right)^2 \left(\frac{N_{II}}{N_I} \right)^2 \\ &= (13.5/0.9)(0.1)^2(\sqrt{10}) = 0.474 \text{ cp} \end{aligned}$$

From Table 1.1-1, we see that this value of ν_{II} corresponds closely to the value for liquid water at 60°C. Thus, the model should operate at 380 rpm, with liquid water at very nearly 60°C.

3A.7 Air entrainment in a draining tank.

As this system is too complex for analytic treatment, we use dimensional analysis. We must establish operating conditions such that both systems satisfy the same dimensionless differential equations and boundary conditions. This means that the large and small systems must be geometrically similar, and that the Froude and Reynolds numbers must be respectively the same for each.

Choose D (tank diameter) as characteristic length, and $(4Q/\pi D^2)$ as characteristic velocity, where Q is the volumetric draw-off rate. Then

$$\text{Re} = \frac{4Q\rho}{\pi D\mu} \quad \text{and} \quad \text{Fr} = \frac{16Q^2}{\pi^2 D^5 g}$$

Subscripts L and S will be used to identify quantities associated with the large and small tanks, respectively. We take the gravitational field g to be the same for both. Then the requirement of equal Reynolds numbers gives

$$\left(\frac{Q_S}{Q_L}\right) \left(\frac{D_L}{D_S}\right) = \left(\frac{\rho_L}{\mu_L}\right) \left(\frac{\mu_S}{\rho_S}\right) = \left(\frac{1.286}{56.7}\right) (1.0037) = 0.02277,$$

and the requirement of equal Froude numbers gives

$$\left(\frac{Q_S}{Q_L}\right) = \left(\frac{D_S}{D_L}\right)^{5/2}$$

Combining these requirements, we obtain

$$\left(\frac{D_S}{D_L}\right) = (0.02277)^{2/3} = 0.080$$

Hence,

$$D_S = (0.080)(60 \text{ ft}) = 4.8 \text{ ft}$$

$$Q_S = (0.080)^{5/2} (800 \text{ gal/min}) = 1.46 \text{ gal/min}$$

Therefore:

- a. The model tank should be 4.8 ft in diameter.
- b. Its draw-off tube should be 0.080 ft in diameter and 0.080 ft high.
- c. Its draw-off tube should have its axis 0.32 ft from the wall of the tank.

Furthermore, if water at 60°F (20°C) is withdrawn from the model tank at 1.46

3B.1 Flow between concentric cylinders and spheres

a. The derivation proceeds as in Example 3.6-3 up to Eq. 3.6-26, which we choose to rewrite as

$$\frac{v_\theta}{r} = D_1 + D_2 \left(\frac{R}{r} \right)^2$$

The boundary conditions are that $v_\theta(\kappa R) = \Omega_i \kappa R$ and $v_\theta(R) = \Omega_o R$. Putting these boundary conditions into the above equation for the angular velocity gives

$$\Omega_i = D_1 + D_2 \frac{1}{\kappa^2} \quad \text{and} \quad \Omega_o = D_1 + D_2$$

These equations can be solved for the integration constants

$$D_1 = \frac{\begin{vmatrix} \Omega_i & 1/\kappa^2 \\ \Omega_o & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1/\kappa^2 \\ 1 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} \Omega_i & 1 \\ \Omega_o & \kappa^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & \kappa^2 \end{vmatrix}} = \frac{\Omega_i \kappa^2 - \Omega_o}{\kappa^2 - 1}$$

$$D_2 = \frac{\begin{vmatrix} 1 & \Omega_i \\ 1 & \Omega_o \end{vmatrix}}{\begin{vmatrix} 1 & 1/\kappa^2 \\ 1 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} \kappa^2 & \Omega_i \\ \kappa^2 & \Omega_o \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & \kappa^2 \end{vmatrix}} = \frac{\kappa^2 (\Omega_o - \Omega_i)}{\kappa^2 - 1}$$

Hence the solution to the differential equation is

$$\frac{v_\theta}{r} = \frac{(\Omega_o - \Omega_i \kappa^2)}{1 - \kappa^2} + \frac{(\Omega_i - \Omega_o)}{1 - \kappa^2} \left(\frac{\kappa R}{r} \right)^2$$

The z-components of the torques on the outer and inner cylinders are

$$T_z = \int_0^L \int_0^{2\pi} (-\tau_{r\theta} R)_{r=R} R d\theta dz = 2\pi L R^2 \left[+\mu r \frac{d}{dr} \left(\frac{v_\theta}{r} \right) \right]_{r=R}$$

$$= 2\pi LR^2 \mu R \frac{(\Omega_i - \Omega_o)}{1 - \kappa^2} (\kappa R)^2 \left(-2 \frac{1}{R^3}\right) = -4\pi\mu L (\Omega_i - \Omega_o) \frac{(\kappa R)^2}{1 - \kappa^2}$$

$$T_z = \int_0^L \int_0^{2\pi} (+\tau_{r\theta} \kappa R)_{r=\kappa R} \kappa R d\theta dz = 2\pi L (\kappa R)^2 \left[-\mu r \frac{d}{dr} \left(\frac{v_\theta}{r} \right) \right]_{r=\kappa R}$$

$$= +4\pi\mu L (\Omega_i - \Omega_o) \frac{(\kappa R)^2}{1 - \kappa^2}$$

b. In Example 3.6-5 it is shown how to get Eq. 3.6-53 for the velocity distribution. The boundary conditions are : $v_\phi(\kappa R) = \kappa R \Omega_i \sin \theta$ and $v_\phi(R) = R \Omega_o \sin \theta$. Equation 3.6-53 can be written in the form

$$\frac{v_\phi}{r \sin \theta} = D_1 + D_2 \left(\frac{R}{r} \right)^3$$

The constants can be obtained according to the method of (a) and the final expression is

$$\frac{v_\phi}{r \sin \theta} = \frac{(\Omega_o - \Omega_i \kappa^3)}{1 - \kappa^3} + \frac{(\Omega_i - \Omega_o)}{1 - \kappa^3} \left(\frac{\kappa R}{r} \right)^3$$

The torques at the outer and inner cylinders are then

$$T_z = \int_0^{2\pi} \int_0^\pi (-\tau_{r\phi})_{r=R} (R \sin \theta) R^2 \sin \theta d\theta d\phi$$

$$= \int_0^{2\pi} \int_0^\pi \left[+\mu r \frac{d}{dr} \left(\frac{v_\phi}{r} \right) \right]_{r=R} (R \sin \theta) R^2 \sin \theta d\theta d\phi$$

$$= -8\pi\mu (\Omega_i - \Omega_o) \frac{(\kappa R)^3}{1 - \kappa^3}$$

$$T_z = \int_0^{2\pi} \int_0^\pi (+\tau_{r\phi})_{r=\kappa R} (\kappa R \sin \theta) (\kappa R)^2 \sin \theta d\theta d\phi = +8\pi\mu (\Omega_i - \Omega_o) \frac{(\kappa R)^3}{1 - \kappa^3}$$

3B.2 Laminar flow in a triangular duct

a. It is clear that the boundary conditions that $v_z = 0$ at $y = H$ and at $y = \pm\sqrt{3}x$. Therefore the no-slip boundary conditions are satisfied. Next it has to be shown that the equation of motion

$$0 = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} \right)$$

is satisfied. Substituting the solution into the second-derivative terms, we get

$$\begin{aligned} \mu \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4\mu LH} \right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (3x^2y - y^3 - 3Hx^2 + Hy^2) \\ = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4LH} \right) (6y - 6H - 6y + 2H) \end{aligned}$$

and this just exactly cancels the pressure-difference term.

b. To get the mass rate of flow we integrate over half the cross-section and multiply by 2:

$$\begin{aligned} w &= 2\rho \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4\mu LH} \right) \int_0^H \int_0^{y/\sqrt{3}} (y - H)(3x^2 - y^2) dx dy \\ &= 2\rho \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4\mu LH} \right) \int_0^H (y - H)(x^3 - y^2x) \Big|_0^{y/\sqrt{3}} dy \\ &= 2\rho \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4\mu LH} \right) \int_0^H (y - H) \left(\frac{-2y^3}{3^{3/2}} \right) dy \\ &= -\rho \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{3^{3/2} \mu LH} \right) \left(-\frac{H^5}{20} \right) = \rho \left(\frac{\sqrt{3}(\mathcal{P}_0 - \mathcal{P}_L)H^4}{180\mu L} \right) \end{aligned}$$

The average velocity is then the volume rate of flow (w/ρ), divided by the cross-sectional area $H^2/\sqrt{3}$ so that

$$\langle v_z \rangle = \frac{(\mathcal{P}_0 - \mathcal{P}_L)H^2}{60\mu L}$$

The maximum velocity will be at the tube center, or at $x = 0$ and $y = 2H/3$, so that

$$v_{z,\max} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)H^2}{27\mu L} = \frac{20}{9}\langle v_z \rangle$$

3B.3 Laminar flow in a square duct

a. The boundary conditions at $x = \pm B$ and $y = \pm B$ are seen to be satisfied by direct substitution into Eq. 3B.3-1. Next we have to see whether the differential equation

$$0 = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} \right)$$

is satisfied. Substituting the derivatives from Eq. 3B.3-1 into this differential equation gives

$$0 = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L} - 2\mu \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{4\mu L} \left(2 - \left(\frac{x}{B} \right)^2 - \left(\frac{y}{B} \right)^2 \right) \right)$$

Hence the differential equation is not satisfied.

b. The expression for the mass flow rate from Eq. 3B.3-1 is given by 4 times the flow rate for one quadrant:

$$\begin{aligned} w &= 4 \cdot \frac{(\mathcal{P}_0 - \mathcal{P}_L) B^2 \rho}{4\mu L} \int_0^B \int_0^B \left[1 - \left(\frac{x}{B} \right)^2 \right] \left[1 - \left(\frac{y}{B} \right)^2 \right] dx dy \\ &= \frac{(\mathcal{P}_0 - \mathcal{P}_L) B^4 \rho}{\mu L} \int_0^1 \int_0^1 (1 - \xi^2)(1 - \eta^2) d\xi d\eta \\ &= \frac{(\mathcal{P}_0 - \mathcal{P}_L) B^4 \rho}{\mu L} \left[\int_0^1 (1 - \xi^2) d\xi \right]^2 \\ &= \frac{(\mathcal{P}_0 - \mathcal{P}_L) B^4 \rho}{\mu L} \left(\frac{2}{3} \right)^2 = \frac{0.444(\mathcal{P}_0 - \mathcal{P}_L) B^4 \rho}{\mu L} \end{aligned}$$

3B.4 Creeping flow between two concentric spheres

a. From Eq. B.4-3, there is only surviving term on the left side

$$\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r v_{\theta} \sin \theta) = 0 \quad \text{whence} \quad v_{\theta} \sin \theta = u(r)$$

b. From Eq. B.6-8 (omitting the left side for *creeping flow*) the only surviving terms are

$$0 = -\frac{1}{r} \frac{d\mathcal{P}}{d\theta} + \mu \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dv_{\theta}}{dr} \right) \right] \quad \text{or} \quad 0 = -\frac{1}{r} \frac{d\mathcal{P}}{d\theta} + \mu \left[\frac{1}{r^2 \sin \theta} \frac{d}{dr} \left(r^2 \frac{du}{dr} \right) \right]$$

c. When the equation in (b) is multiplied by $r \sin \theta$, the left side is a function of θ alone, and the right side contains only r . This means that both sides must be equal to some constant, which we call B . This gives Eqs. 3B.4-2 and 3.

Integration of the pressure equation proceeds as follows:

$$\int_{\mathcal{P}_1}^{\mathcal{P}_2} d\mathcal{P} = \int_{\varepsilon}^{\pi-\varepsilon} \frac{d\theta}{\sin \theta} \quad \text{or} \quad \mathcal{P}_2 - \mathcal{P}_1 = B \ln \frac{\tan \frac{1}{2}(\pi - \varepsilon)}{\tan \frac{1}{2} \varepsilon} = B \ln \frac{\cot \frac{1}{2} \varepsilon}{\tan \frac{1}{2} \varepsilon}$$

From this we get the constant B

$$B = \frac{\mathcal{P}_2 - \mathcal{P}_1}{\ln \cot^2 \frac{1}{2} \varepsilon} = \frac{\mathcal{P}_2 - \mathcal{P}_1}{2 \ln \cot \frac{1}{2} \varepsilon}$$

Next we integrate the velocity equation

$$\frac{d}{dr} \left(r^2 \frac{du}{dr} \right) = \frac{Br}{\mu} \quad \text{or} \quad u = \frac{BR}{2\mu} \left(\frac{r}{R} - \frac{C_1 R}{r} + C_2 \right)$$

where we have selected the constants of integration in such a way that they will be dimensionless: $C_1 = -\kappa$ and $C_2 = -\kappa - 1$ (from the no-slip condition at the walls). When this solution is combined with the expression for B we get:

$$u = \frac{BR}{2\mu} \left[-\left(1 - \frac{r}{R}\right) - \kappa \left(1 - \frac{R}{r}\right) \right] = \frac{\mathcal{P}_1 - \mathcal{P}_2}{2 \ln \cot \frac{1}{2} \varepsilon} \cdot \frac{R}{2\mu} \left[\left(1 - \frac{r}{R}\right) + \kappa \left(1 - \frac{R}{r}\right) \right]$$

which leads to Eq. 3B.4-5.

d. The mass rate of flow must be the same through any cross-section. It is easiest to get w at $\theta = \frac{1}{2}\pi$ where $v_\theta = u(r)/\sin\theta = u(r)/\sin\frac{1}{2}\pi = u(r)$:

$$\begin{aligned}
 w &= \int_0^{2\pi} \int_{\kappa R}^R \rho v_\theta \Big|_{\theta=\frac{1}{2}\pi} r dr d\phi = 2\pi R^2 \rho \int_\kappa^1 u \xi d\xi \\
 &= 2\pi R^2 \rho \cdot \frac{(\mathcal{P}_1 - \mathcal{P}_2)R}{4\mu \ln \cot \frac{1}{2}\varepsilon} \int_\kappa^1 \left[(1-\xi) + \kappa \left(1 - \frac{1}{\xi} \right) \right] \xi d\xi \\
 &= 2\pi R^2 \rho \cdot \frac{(\mathcal{P}_1 - \mathcal{P}_2)R}{4\mu \ln \cot \frac{1}{2}\varepsilon} \cdot \frac{1}{6} (1-\kappa)^3 \quad (\text{see Eq. 3B.4-6})
 \end{aligned}$$

3B.5 Parallel-disk viscometer

a. The equation of continuity (Eq. B.6-5) just gives $0 = 0$. Equation B.6-5 gives for the θ -component of the equation of motion

$$0 = \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (rv_\theta) \right) + \frac{\partial^2 v_\theta}{\partial z^2}$$

b. When the postulated form for the tangential component of the velocity is inserted into this, we get simply $d^2 f/dz^2 = 0$, which has to be solved with the boundary conditions that $f(0) = 0$ and $f(B) = \Omega$, which are just statements of the no-slip conditions on the wetted surfaces of the disks. It is easily shown that $f = \Omega z/B$, and therefore that $v_\theta = \Omega r z/B$.

c. The z -component of the torque exerted on the fluid by the upper rotating disk is

$$T_z = \int_0^{2\pi} \int_0^R (-\tau_{z\theta} r) \Big|_{z=B} r dr d\theta = 2\pi \int_0^R \left[+\mu \left(\frac{\partial v_\theta}{\partial z} \right) r \right] \Big|_{z=B} r dr = 2\pi\mu \cdot \frac{\Omega}{B} \int_0^R r^3 dr$$

We can now do the r -integration and solve for the viscosity to get the desired formula, $\mu = 2BT_z/\pi\Omega R^4$.

3B.6 Circulating flow in an annulus

a. Neglect of curvature gives for the z-component of the equation of motion

$$0 = -\frac{d\mathcal{P}}{dz} + \mu \frac{d^2 v_z}{dr^2} \quad \text{or} \quad \mu \frac{d^2 v_z}{dr^2} = \frac{d\mathcal{P}}{dz}$$

The left side is a function of r alone and the right side is a function of z alone, so that both sides must equal a constant. Therefore

$$\frac{d^2 v_z}{dr^2} = C' \quad \text{with boundary conditions } v_z(\kappa R) = v_0 \text{ and } v_z(R) = 0$$

Use the dimensionless variables $\phi = v_z/v_0$ and $\xi = r/R$; instead of the latter, it is more convenient to use $\zeta = (\xi - \kappa)/(1 - \kappa)$ in this part of the problem. Then

$$\frac{d^2 \phi}{d\zeta^2} = 2C_1 \quad \text{with boundary conditions } \phi(0) = 1 \text{ and } \phi(1) = 0$$

Integration of this equation gives

$$\phi = C_1 \zeta^2 + C_2 \zeta + C_3$$

The boundary conditions give $C_3 = 1$ and $C_2 = -(1 + C_1)$ so that

$$\phi = C_1 \zeta^2 - (1 + C_1) \zeta + 1$$

The mass-balance condition $\int_0^1 \phi d\zeta = 0$ gives $C_1 = 3$, and the velocity distribution is

$$\phi = 3\zeta^2 - 4\zeta + 1$$

b. We use exactly the same procedure when the thin-slit approximation is not made, but the algebraic manipulations are messier. The equation to be solved is

$$\mu \frac{1}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) = \frac{d\mathcal{P}}{dz}$$

with the same no-slip boundary conditions as before. Use the same dimensionless variables as in (a) and find that the solution has the form

$$\phi = C_1 \xi^2 + C_2 \ln \xi + C_3$$

The three constants of integration are determined from the no-slip boundary conditions $\phi(\kappa) = 1$ and $\phi(1) = 0$, along with the mass-conservation condition $\int_0^{2\pi} \int_{\kappa}^1 \phi \xi d\xi d\theta = 0$. The results are

$$C_1 = -C_3 = -\frac{1 - [2\kappa^2 / (1 - \kappa^2)] \ln(1/\kappa)}{(1 - \kappa^2) - (1 + \kappa^2) \ln(1/\kappa)}; \quad C_2 = \frac{1 - \kappa^2}{(1 - \kappa^2) - (1 + \kappa^2) \ln(1/\kappa)}$$

With these expressions for the integration constants, Eq. 3B.6-2 follows.

3B.7 Momentum fluxes for creeping flow into a slot

a. Inside the slot, the nonzero component of $\rho v v$ is

$$\rho v_x v_x = \rho \left(\frac{3w}{4BW\rho} \right)^2 \left[1 - \left(\frac{y}{B} \right)^2 \right]^2$$

Outside the slot, the nonzero component of $\rho v v$ are

$$\rho v_x v_x = \rho \left(\frac{2w}{\pi W\rho} \right)^2 \frac{x^6}{(x^2 + y^2)^4}; \quad \rho v_y v_y = \rho \left(\frac{2w}{\pi W\rho} \right)^2 \frac{x^4 y^2}{(x^2 + y^2)^4}$$

$$\rho v_x v_y = \rho v_y v_x = \rho \left(\frac{2w}{\pi W\rho} \right)^2 \frac{x^5 y}{(x^2 + y^2)^4}$$

b. At $x = -a, y = 0$

$$\rho v_x v_x = \rho \left(\frac{2w}{\pi W\rho} \right)^2 \frac{1}{a^2}$$

This quantity is positive as we would expect, since positive x -momentum is being transported in the x -direction.

c. At $x = -a, y = +a$.

$$\rho v_x v_y = -\rho \left(\frac{2w}{\pi W\rho} \right)^2 \frac{1}{2^4 a^2}$$

This is negative, since at the point in question the y -momentum is negative and being transported in the x -direction.

d. The total flow of kinetic energy in the slot is (if we use $\eta = y/B$):

$$\begin{aligned} \int_0^W \int_{-B}^{+B} \left(\frac{1}{2} \rho v^2 \right) v_x dy dz &= 2W \int_0^B \left(\frac{1}{2} \rho v_x^3 \right) dy = \rho B W \left(\frac{3w}{4BW\rho} \right) \int_0^1 (1 - \eta^3) d\eta \\ &= \frac{16}{35} \rho B W \left(\frac{3w}{4BW\rho} \right)^3 \end{aligned}$$

The total flow of kinetic energy across the plane at $x = -a$ is

$$\begin{aligned}
\int_0^W \int_{-\infty}^{+\infty} (\frac{1}{2}\rho v^2) v_x dy dz &= \frac{1}{2}\rho W \int_{-\infty}^{+\infty} (v_x^2 + v_y^2) v_x dy \\
&= \frac{1}{2}\rho W \left(\frac{2w}{\pi W \rho} \right)^3 \int_{-\infty}^{+\infty} \left[\frac{a^9}{(a^2 + y^2)^6} + \frac{a^7 y^2}{(a^2 + y^2)^6} \right] dy \\
&= \frac{105}{768} \rho W \left(\frac{2w}{\pi W \rho} \right)^3 \frac{\pi}{a^2}
\end{aligned}$$

The integrals appearing here can be found in integral tables.

We conclude that the total flow of kinetic energy across the plane at $x = -a$ is not the same as that in the slit. As $a \rightarrow \infty$, the flow of kinetic energy tends toward zero, since the fluid velocity tends to zero as $x \rightarrow -\infty$. This emphasizes that kinetic energy is not conserved.

e. Eq. 3B.7-1 clearly satisfies the equation of continuity, since for incompressible flow $(\partial v_x / \partial x) + (\partial v_y / \partial y) + (\partial v_z / \partial z) = 0$. When the derivatives are calculated from Eqs. 3B.7-2, 3, and 4, it is found that these expressions also satisfy the incompressible equation of continuity as well.

f. From Eqs. 3B.7-2 and B.1-1 we get

$$\tau_{xx} = -2\mu \left. \frac{\partial v_x}{\partial x} \right|_{y=0} = 2\mu \left(\frac{2w}{\pi W \rho} \right) \left[\frac{3x^2}{(x^2 + y^2)^2} - \frac{4x^4}{(x^2 + y^2)^3} \right]_{y=0} = -\frac{4w\mu}{\pi W \rho x^2}$$

$$\tau_{xx} = -2\mu \left. \frac{\partial v_x}{\partial x} \right|_{x=0} = 2\mu \left(\frac{2w}{\pi W \rho} \right) \left[\frac{3x^2}{(x^2 + y^2)^2} - \frac{4x^4}{(x^2 + y^2)^3} \right]_{x=0} = 0$$

The second of these is an illustration of Example 3.1-1.

g. From Eqs. 3B.7-2, 3B.7-3, and B.1-4, we get, after evaluating the derivatives

$$\tau_{yx} = -\mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \Big|_{x=0} = 0$$

3B.8 Velocity distribution for creeping flow toward a slot

a. For the given postulates, the equation of continuity gives

$$\frac{1}{r} \frac{\partial}{\partial r}(rv_r) = 0 \quad \text{from which it follows that } v_r = \frac{1}{r} f(\theta)$$

Since the flow is symmetric about $\theta = 0$, $df/d\theta = 0$ at $\theta = 0$; and since the fluid velocity is zero at $\theta = \pm \frac{1}{2}\pi$, it follows that $f = 0$ at $\theta = \pm \frac{1}{2}\pi$.

b. The components of the equation of motion given in Eqs. B.6-4 and 5, appropriately simplified are

$$0 = -\frac{\partial \mathcal{P}}{\partial r} + \frac{\mu}{r^3} \frac{d^2 f}{d\theta^2} \quad \text{and} \quad 0 = -\frac{\partial \mathcal{P}}{\partial \theta} + \frac{2\mu}{r^2} \frac{df}{d\theta}$$

c. When the first equation is differentiated with respect to θ and the second with respect to r and the two results subtracted we get Eq. 3B.8-1.

d. The equation in Eq. 3B.8-1 can be integrated once to get

$$\frac{d^2 f}{d\theta^2} + 4f = C_1$$

A particular integral is $f_{PI} = \frac{1}{4}C_1$, and the complementary function is (according to Eq. C.1-3) $f_{CF} = C_2 \cos 2\theta + C_3 \sin 2\theta$. The complete solution is then the sum of these two functions.

e. The integration constants are determined from the boundary conditions. It is found that $C_3 = 0$, and that $\frac{1}{4}C_1 = C_2$. Then from

$$w = -W\rho \int_{-\pi/2}^{+\pi/2} v_r r d\theta = -W\rho \int_{-\pi/2}^{+\pi/2} f d\theta = -2W\rho C_2 \int_{-\pi/2}^{+\pi/2} \cos^2 \theta d\theta = -W\rho C_2 \pi$$

we get $C_2 = -w/W\rho\pi$ and the velocity distribution is given by Eq. 3B.8-2.

f. From the velocity distribution and the equations obtained in (b) we can get

$$\frac{\partial \mathcal{P}}{\partial r} = \frac{\mu}{r^3} \left(\frac{4w}{\pi W \rho} \right) (\cos^2 \theta - \sin^2 \theta)$$

and hence

$$\mathcal{P} = -\frac{\mu}{r^2} \left(\frac{2w}{\pi W \rho} \right) (\cos^2 \theta - \sin^2 \theta) + F(\theta) \quad (*)$$

Furthermore

$$\frac{\partial \mathcal{P}}{\partial \theta} = \frac{2\mu}{r^2} \frac{df}{d\theta} \quad \text{or} \quad \mathcal{P} = -\left(\frac{2\mu}{r^2} \right) \left(\frac{2w}{\pi W \rho} \right) \cos^2 \theta + G(r)$$

Here F and G are arbitrary functions of their arguments. The second expression for the modified pressure can be rewritten as

$$\begin{aligned} \mathcal{P} &= -\frac{\mu}{r^2} \left(\frac{2w}{\pi W \rho} \right) (\cos^2 \theta + (1 - \sin^2 \theta)) + G(r) \\ &= -\frac{\mu}{r^2} \left(\frac{2w}{\pi W \rho} \right) (\cos^2 \theta - \sin^2 \theta) + H(r) \quad (*) \end{aligned}$$

By comparing the two (*) expressions for the modified pressure, we see that they are the same except for the functions F and H . Since the first is a function of θ alone and the second a function of r alone, they must both be equal to a constant, which we call \mathcal{P}_∞ . This is the value of the modified pressure at $r = \infty$.

g. The total normal stress exerted on the wall at $\theta = \pi/2$ is (when one uses the result of Example 3.1-1)

$$\begin{aligned} \pi_{\theta\theta}|_{\theta=\pi/2} &= (p + \tau_{\theta\theta})|_{\theta=\pi/2} = (\mathcal{P} - \rho gh)|_{\theta=\pi/2} \\ &= \mathcal{P}_\infty + \frac{2\mu w}{\pi W \rho r^2} - \rho gh = p_\infty + \frac{2\mu w}{\pi W \rho r^2} \end{aligned}$$

h. From Eq. B.1-11

$$\begin{aligned}\tau_{\theta r}|_{\theta=\pi/2} &= -\mu \left[r \frac{\partial}{\partial r} \left(\frac{v_{\theta}}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]_{\theta=\pi/2} \\ &= -\mu \left[0 - \left(\frac{2w}{\pi W \rho r^2} \right) (-2 \cos \theta \sin \theta) \right]_{\theta=\pi/2} = 0\end{aligned}$$

The first term is zero since $v_{\theta}/r = f$ since $v_{\theta} = 0$ was one of the postulates. The second term is zero, as can be seen by using Eq. 3B.8-2, and the fact that $\cos \frac{\pi}{2} = 0$. This is agreement with the result in Problem 3B.7(g).

i. Since the z -component of the velocity is zero, we can expand the velocity vector in either the cylindrical coordinate system or the Cartesian system thus

$$\mathbf{v} = \delta_r v_r + \delta_{\theta} v_{\theta} = \delta_x v_x + \delta_y v_y$$

Since $v_{\theta} = 0$ was one of the postulates, when we take the dot product of this equation with δ_x , we get the x -component of the velocity

$$v_x = (\delta_x \cdot \delta_r) = v_r \cos \theta = -\frac{2w}{\pi W \rho r} \cos^3 \theta = -\frac{2wx^3}{\pi W \rho r^4} = -\frac{2wx^3}{\pi W \rho (x^2 + y^2)^2}$$

Similarly for the y -component of the velocity

$$\begin{aligned}v_y &= (\delta_y \cdot \delta_r) = v_r \sin \theta = -\frac{2w}{\pi W \rho r} \cos^2 \theta \sin \theta = -\frac{2wx^2 y}{\pi W \rho r^4} \\ &= -\frac{2wx^2 y}{\pi W \rho (x^2 + y^2)^2}\end{aligned}$$

These results are in agreement with Eqs. 3B.7-2 and 3.

3B.9 Slow transverse flow around a cylinder

a. At the cylinder surface we get by using Eq. B.1-11

$$p|_{r=R} = p_\infty - \frac{C\mu v_\infty \cos \theta}{R} - \rho g R \sin \theta$$

$$\tau_{r\theta}|_{r=R} = -\mu \left(r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right) \Big|_{r=R} = \frac{C\mu v_\infty \sin \theta}{R}$$

Also from Example 3.1-1 we know that $\tau_{rr}|_{r=R} = 0$.

b. If \mathbf{n} is the outwardly directed unit normal vector for the cylinder, then the force per unit area acting on the surface is $-\mathbf{n} \cdot \boldsymbol{\pi}$ evaluated at the surface. But $\mathbf{n} = \boldsymbol{\delta}_r$, so that the x -component of the force per unit area at every point is

$$-(\boldsymbol{\delta}_x \cdot [\boldsymbol{\delta}_r \cdot \boldsymbol{\pi}])|_{r=R} = -(\boldsymbol{\delta}_x \cdot [\boldsymbol{\delta}_r \cdot (p\boldsymbol{\delta} + \boldsymbol{\tau})])|_{r=R}$$

The pressure and stress terms are evaluated thus (using Eq. A.6-13):

$$-(\boldsymbol{\delta}_x \cdot [\boldsymbol{\delta}_r \cdot p\boldsymbol{\delta}])|_{r=R} = -(\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_r p)|_{r=R} = -p|_{r=R} \cos \theta$$

$$\begin{aligned} -(\boldsymbol{\delta}_x \cdot [\boldsymbol{\delta}_r \cdot \boldsymbol{\tau}])|_{r=R} &= -(\boldsymbol{\delta}_x \cdot [\boldsymbol{\delta}_r \cdot (\cdots + \boldsymbol{\delta}_r \boldsymbol{\delta}_\theta \tau_{r\theta} + \cdots)])|_{r=R} \\ &= -(\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_\theta) \tau_{r\theta}|_{r=R} = \tau_{r\theta}|_{r=R} \sin \theta \end{aligned}$$

These expressions lead to Eq. 3B.9-5.

c. The total force on a length L of the cylinder is then

$$\begin{aligned} F_x &= \int_0^L \int_0^{2\pi} (-p|_{r=R} \cos \theta + \tau_{r\theta}|_{r=R} \sin \theta) R d\theta dz \\ &= RL \int_0^{2\pi} \left(-p_\infty \cos \theta + \frac{C\mu v_\infty \cos^2 \theta}{R} + \rho g R \sin \theta \cos \theta + \frac{C\mu v_\infty \sin^2 \theta}{R} \right) d\theta \\ &= RL \left(\frac{C\mu v_\infty}{R} \right) \int_0^{2\pi} d\theta = 2\pi C\mu v_\infty L \end{aligned}$$

The first and third integrals in the next-to-last line are zero since the integrands are odd.

3B.10 Radial flow between parallel disks

a. The continuity equation is, for $v_r = v_r(r, z)$, from Eq. B.4-2

$$\frac{1}{r} \frac{\partial}{\partial r} (rv_r) = 0 \quad \text{from which} \quad rv_r = \phi(z)$$

The equation of motion is obtained from Eq. B.6-4

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{d\mathcal{P}}{dr} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (rv_r) \right) + \frac{\partial^2 v_r}{\partial z^2} \right]$$

b. When the results for the equation of continuity in (a) are used in the equation of motion, we get Eq. 3B.10-1.

c. With the creeping flow assumption, Eq. 3B.10-1 gives

$$r \frac{d\mathcal{P}}{dr} = \mu \frac{d^2 \phi}{dz^2} \quad \text{from which} \quad r \frac{d\mathcal{P}}{dr} = B \quad \text{and} \quad \mu \frac{d^2 \phi}{dz^2} = B$$

since the left side is a function of r alone and the right side a function of z alone, and therefore both sides must be equal to a constant, B . When the pressure equation is integrated from r_1 to r_2 , we get

$$\int_{\mathcal{P}_1}^{\mathcal{P}_2} d\mathcal{P} = B \int_{r_1}^{r_2} \frac{dr}{r} \quad \text{whence} \quad B = \frac{\mathcal{P}_2 - \mathcal{P}_1}{\ln(r_2/r_1)}$$

d. When this result is substituted into the ϕ equation we get

$$\frac{d^2 \phi}{dz^2} = -\frac{\mathcal{P}_1 - \mathcal{P}_2}{\mu \ln(r_2/r_1)} \quad \text{from which} \quad \phi = -\frac{\mathcal{P}_1 - \mathcal{P}_2}{\mu \ln(r_2/r_1)} \frac{z^2}{2} + C_1 z + C_2$$

The integration constants are obtained from applying two boundary conditions. We could require that $\phi = 0$ at $z = \pm b$, and thereby determine the integration constants. Another method is to recognize that the flow is symmetric about $z = 0$, and use as one of the boundary conditions $d\phi/dz = 0$ at $z = 0$. Either method will give $C_1 = 0$, and then C_2 is easily obtained. The final result is

$$\phi = \frac{(\mathcal{P}_1 - \mathcal{P}_2)b^2}{2\mu \ln(r_2/r_1)} \left[1 - \left(\frac{z}{b} \right)^2 \right]$$

Division by r then gives Eq. 3B.10-3.

e. The mass rate of flow at any cylindrical surface in the system must be the same. Select the surface at $r = r_2$ and obtain

$$w = \int_0^{2\pi} \int_{-b}^{+b} \rho v_r \Big|_{r=r_2} dz r_2 d\theta = 2\pi \rho b \cdot 2 \cdot \frac{(\mathcal{P}_1 - \mathcal{P}_2)b^2}{2\mu \ln(r_2/r_1)} \int_0^1 (1 - \xi^2) d\xi$$

The integral gives 2/3, so that Eq. 3B.10-4 is obtained.

3B.11 Radial flow between two coaxial cylinders

a. From Eq. B.4-2 we get for this flow, with $v_r(r)$,

$$\frac{1}{r} \frac{d}{dr}(rv_r) = 0 \quad \text{whence} \quad v_r = \frac{C}{r} \quad \text{where } C \text{ is a constant}$$

At $r = R$, $v_r(R) = C/R$ so that $C = Rv_r(R)$.

b. The relations in Eq. 3B.11-1 follow immediately from Eqs. B.6-4, 5, and 6 for the velocity profile $v_r(r)$ in (a).

c. Integration from r to R gives

$$\mathcal{P}(R) - \mathcal{P}(r) = \rho C^2 \left(-\frac{1}{2r^2} \right) \Big|_r^R = -\frac{1}{2} \rho C^2 \left(\frac{1}{R^2} - \frac{1}{r^2} \right)$$

This gives, making use of the meaning of C obtained in (a)

$$\mathcal{P}(r) - \mathcal{P}(R) = \frac{1}{2} \rho [Rv_r(R)]^2 \left(\frac{1}{R^2} - \frac{1}{r^2} \right) = \frac{1}{2} \rho [v_r(R)]^2 \left[1 - \left(\frac{R}{r} \right)^2 \right]$$

d. The only nonzero components are (from Eqs. B.1-8 to 13)

$$\tau_{rr} = -2\mu \frac{dv_r}{dr} = +2\mu C \frac{1}{r^2}; \quad \tau_{\theta\theta} = -2\mu \frac{v_r}{r} = -2\mu C \frac{1}{r^2}$$

3B.12 Pressure distribution in incompressible fluids

The equation of motion method to get the pressure distribution is correct. On the other hand, the second method gives nonsense, as one can see from Fig. 3.5-1. For an incompressible fluid (the vertical straight line), specifying the density does not give any information about the pressure.

3B.13 Flow of a fluid through a sudden contraction

a. For an incompressible fluid, Eq. 3.5-12 becomes

$$\frac{1}{2}\rho(v_2^2 - v_1^2) + (\rho - \rho) + \rho g(h_2 - h_1) = 0 \quad \text{or} \quad \frac{1}{2}\rho(v_2^2 - v_1^2) + (\rho - \rho) = 0$$

If "1" is the large tube and "2" the small one, then the fluid velocity in "2" must be greater than in "1." Then the modified pressure in "2" must be less than that in "1." Thus the modified pressure decreases as the fluid moves from the large cross-section region to the small cross-section region, in agreement with experimental observations.

b. For an ideal gas, Eq. 3.5-12 becomes

$$\frac{1}{2}\rho(v_2^2 - v_1^2) + \frac{RT}{M} \ln \frac{p_2}{p_1} + \rho g(h_2 - h_1) = 0$$

The pressure and elevation terms may no longer be combined. If the elevation does not change, the pressure decreases as the fluid moves into the contracted part of the tube.

3B.14 Torricelli's equation for efflux from a tank

From Eq. 3.5-12 we get

$$\frac{1}{2}(v_{\text{efflux}}^2 - 0) + \frac{1}{\rho}(p_{\text{atm}} - p_{\text{atm}}) + g(0 - h) = 0$$

Here it has been assumed that the velocity at the surface is virtually zero, that the pressure is atmospheric at both "1" and "2", and that the datum plane for the height is at the exit tube. When the above equation is solved for the efflux velocity we get Torricelli's equation.

3B.15 Shape of free surface in tangential annular flow

a. The velocity distribution is given by Eq. 3.6-32, and the equations corresponding to Eqs. 3.6-38 and 39 are:

$$\frac{\partial p}{\partial r} = \rho \left(\frac{\Omega_i \kappa^2 R}{1 - \kappa^2} \right)^2 \frac{1}{r} \left[\left(\frac{R}{r} \right)^2 - 2 + \left(\frac{r}{R} \right)^2 \right] \quad \text{and} \quad \frac{\partial p}{\partial z} = -\rho g$$

Integration of these equations gives (see Eqs. 3.6-40, 41, and 42)

$$p = \frac{1}{2} \rho \left(\frac{\Omega_i \kappa^2 R}{1 - \kappa^2} \right)^2 \left[-\left(\frac{R}{r} \right)^2 - 4 \ln r + \left(\frac{r}{R} \right)^2 \right] - \rho g z + C$$

Now let $p = p_{\text{atm}}$ at $r = R$ and $z = z_R$, where z_R is the height of the liquid at the outer-cylinder surface. Then we can write at $r = R$ and $z = z_R$

$$p_{\text{atm}} = \frac{1}{2} \rho \left(\frac{\Omega_i \kappa^2 R}{1 - \kappa^2} \right)^2 [-1 - 4 \ln R + 1] - \rho g z_R + C$$

which is the equation that determines C . When we subtract the last equation from the equation for p , we get

$$p - p_{\text{atm}} = \frac{1}{2} \rho \left(\frac{\Omega_i \kappa^2 R}{1 - \kappa^2} \right)^2 \left(-\frac{1}{\xi^2} - 4 \ln \xi + \xi^2 \right) - \rho g (z - z_R)$$

The equation of the liquid surface is then the locus of all points for which $p = p_{\text{atm}}$, or

$$z_R - z = \frac{1}{2g} \left(\frac{\Omega_i \kappa^2 R}{1 - \kappa^2} \right)^2 \left(\frac{1}{\xi^2} + 4 \ln \xi - \xi^2 \right)$$

b. When the outer cylinder is rotating, we can use Eq. 3.6-29 for the velocity distribution to get

$$p = \rho \left(\frac{\Omega_o \kappa R}{1 - \kappa^2} \right)^2 \left[\frac{1}{2} \left(\frac{r}{\kappa R} \right)^2 - 2 \ln r - \frac{1}{2} \left(\frac{\kappa R}{r} \right)^2 \right] - \rho g z + C$$

Then, we select $r = R$ and $z = z_R$ as the point to determine C .

$$p_{\text{atm}} = \rho \left(\frac{\Omega_o \kappa R}{1 - \kappa^2} \right)^2 \left[\frac{1}{2} \left(\frac{1}{\kappa} \right)^2 - 2 \ln R - \frac{1}{2} (\kappa)^2 \right] - \rho g z_R + C$$

Subtracting, we get

$$p - p_{\text{atm}} = \frac{\rho}{2} \left(\frac{\Omega_o \kappa R}{1 - \kappa^2} \right)^2 \left[\left(\frac{\xi}{\kappa} \right)^2 - \left(\frac{1}{\kappa} \right)^2 - 4 \ln \xi - \left(\frac{\kappa}{\xi} \right)^2 + (\kappa)^2 \right] + \rho g (z_R - z)$$

From this equation we can get Eq. 3B.15-2 by setting the left side of the equation equal to zero and solving for $z_R - z$.

3B.16 Flow in a slit with uniform cross flow

From Eq. B.6-1, for this problem we have

$$\rho v_0 \frac{dv_x}{dy} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L} + \mu \frac{d^2 v_x}{dy^2} \quad \text{or} \quad \frac{d^2 \phi}{d\eta^2} - A \frac{d\phi}{d\eta} + 1 = 0$$

in which $\eta = y/B$, $\phi = v_x / [(\mathcal{P}_0 - \mathcal{P}_L)B^2/\mu L]$, and $A = Bv_0\rho/\mu$. This equation has to be solved with the no-slip condition at $\eta = 0, 1$. We write the solution as the sum of a complementary function and a particular integral. The equation for the complementary function is

$$\frac{d^2 \phi_{CF}}{d\eta^2} - A \frac{d\phi_{CF}}{d\eta} = 0 \quad \text{with solution} \quad \phi_{CF} = \frac{C_1}{A} e^{A\eta} + C_2$$

By inspection, the particular integral is $\phi_{PI} = \eta/A$. Application of the boundary conditions then gives the constants of integration. The final solution is then (with $A = Bv_0\rho/\mu$)

$$\phi = \frac{1}{A} \left(\eta - \frac{e^{A\eta} - 1}{e^A - 1} \right) \quad \text{or} \quad v_x = \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{\mu L} \frac{1}{A} \left(\frac{y}{B} - \frac{e^{Ay/B} - 1}{e^A - 1} \right)$$

b. The mass rate of flow in the x -direction is then

$$\begin{aligned} w &= \int_0^W \int_0^B \rho v_x dy dz = \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^3 W \rho}{\mu L} \int_0^1 \phi d\eta \\ &= \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^3 W \rho}{\mu L} \frac{1}{A} \left(\frac{1}{2} - \frac{1}{A} + \frac{1}{e^A - 1} \right) \end{aligned}$$

c By making a Taylor-series expansion about $A = 0$, from (a) we get $\phi = \frac{1}{2}(\eta - \eta^2) + O(A)$. When $A \rightarrow 0$, this result can be shown to be equivalent to Eq. 2B.3-2. Similarly, A Taylor-series expansion about $A = 0$ yields from the result in (b)

$$\frac{w}{[(\mathcal{P}_0 - \mathcal{P}_L)WB^3\rho/\mu L]} = \frac{1}{A} \left(\frac{1}{2} - \frac{1}{A} + \frac{1}{e^A - 1} \right)$$

$$\begin{aligned}
&= \frac{1}{A} \left(\frac{1}{2} - \frac{1}{A} + \frac{1}{\left(1 + A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + \dots\right) - 1} \right) \\
&= \frac{1}{A} \left(\frac{1}{2} - \frac{1}{A} + \frac{1}{A \left(1 + \frac{1}{2}A + \frac{1}{3}A^2 + \dots\right)} \right) \\
&= \frac{1}{A} \left(\frac{1}{2} - \frac{1}{A} + \frac{1 - \frac{1}{2}A + \frac{1}{12}A^2 + \dots}{A} \right) \\
&= \frac{1}{12} + O(A)
\end{aligned}$$

But the "B" in this problem is twice the "B" of problem 2B.3. If we switch to the "B" of Problem 2B.3, it is found that the answers agree exactly.

d. For the coordinate system here, we select as the dimensionless quantities

$$Y = \frac{y}{b}; \quad V = \frac{v_x}{(\mathcal{P}_0 - \mathcal{P}_L)b^2/\mu L}; \quad a = \frac{bv_0\rho}{\mu}$$

Then the differential equation and boundary conditions are

$$a \frac{dV}{dY} = 1 + \frac{d^2V}{dY^2} \quad \text{with} \quad V(\pm 1) = 0$$

The solution is then the sum of a complementary function and a particular integral (as before in (c))

$$V = \frac{C_1}{a} e^{aY} + C_2 + \frac{Y}{a}$$

Application of the boundary conditions then leads to

$$V = \frac{-e^{aY} + \cosh a + Y \sinh a}{a \sinh a}$$

Then the average value of this over the cross-section of flow is

$$\langle V \rangle = \frac{\int_{-1}^{+1} V dY}{\int_{-1}^{+1} dY} = \frac{1}{2} \frac{\int_{-1}^{+1} (-e^{aY} + \cosh a + Y \sinh a) dY}{a \sinh a} = \frac{-(1/a) \sinh a + \cosh a}{a \sinh a}$$

Then we can form the ratio given in Eq. 3B.16-3:

$$\frac{V}{\langle V \rangle} = \frac{e^{aY} - Y \sinh a - \cosh a}{(1/a) \sinh a - \cosh a}$$

As a check on this we can go directly from Eqs. 3B.16-1 and 2 to Eq. 3B.16-3. From the first two equations we get

$$\frac{v_x}{\langle v_x \rangle} = \frac{v_x}{w/WB} = \frac{\frac{y}{B} - \frac{e^{Ay/B} - 1}{e^A - 1}}{\frac{1}{2} - \frac{1}{A} + \frac{1}{e^A - 1}} = \frac{2A[(e^A - 1)y - B(e^{Ay/B} - 1)]}{B[(A - 2)(e^A - 1) + 2A]}$$

Next, we make the connections between the notations in the two different approaches:

$$y = z + b; \quad B = 2b; \quad A = 2\alpha$$

(the "y" of part (c) is called "z" here, and $\zeta = z/b$). Then

$$\begin{aligned} \frac{v_x}{\langle v_x \rangle} &= \frac{\alpha[(e^{2\alpha} - 1)(z + b) - 2b(e^{(\alpha/b)(z+b)} - 1)]}{b[(\alpha - 1)(e^{2\alpha} - 1) + 2\alpha]} \\ &= \frac{(e^{2\alpha} - 1)(\zeta + 1) - 2(e^{\alpha\zeta} e^\alpha - 1)}{(1/\alpha)[(\alpha - 1)(e^{2\alpha} - 1) + 2\alpha]} = \frac{(e^\alpha - e^{-\alpha})(\zeta + 1) - 2(e^{\alpha\zeta} - e^{-\alpha})}{(1/\alpha)[(\alpha - 1)(e^\alpha - e^{-\alpha}) + 2\alpha e^{-\alpha}]} \\ &= \frac{\zeta(e^\alpha - e^{-\alpha}) + (e^\alpha + e^{-\alpha}) - 2e^{\alpha\zeta}}{(1/\alpha)[\alpha(e^\alpha + e^{-\alpha}) - (e^\alpha - e^{-\alpha})]} = \frac{\zeta \sinh \alpha + \cosh \alpha - e^{\alpha\zeta}}{\cosh \alpha - (1/\alpha) \sinh \alpha} \end{aligned}$$

3C.1 Parallel-disk compression viscometer

a. The equation of continuity of Eq. B.4-2 for incompressible fluids, taking into account the symmetry about the z -axis is just Eq. 3C.1-6. The equation of motion in Eq. 3C.1-7 comes from Eq. B.6-4 ignoring the hydrostatic pressure, the inertial force terms, and omitting the terms that are small.

b. Equation 3C.1-7 can be integrated with respect to z to give

$$v_r = \frac{1}{2\mu} \frac{dp}{dr} z^2 + C_1 z + C_2$$

The constant C_2 is found to be zero from the boundary condition in Eq. 3C.1-8, and C_1 is found from Eq. 3C.1-9.

c. Integrating of Eq. 3C.1-6 with respect to z from 0 to H

$$\int_0^H \frac{1}{r} \frac{\partial}{\partial r} \left(r \cdot \frac{1}{2\mu} \frac{dp}{dr} z(z-H) \right) dz = - \int_0^{-v_0} dv_z$$

Performing the integrations then gives

$$-\frac{H^3}{12\mu} \frac{1}{r} \frac{d}{dr} \left(r \frac{dp}{dr} \right) = v_0$$

d. Integration of the equation in (c) then yields

$$p = -\frac{12\mu v_0}{H^3} \frac{r^2}{4} + C_1 \ln r + C_2$$

The integration constant C_1 must be zero, since the pressure is finite at the center of the disks, and C_2 is determined from Eq. 3C.1-10. Equation 3C.1-13 is thus obtained.

e. The force on the upper plate is then

$$F(t) = \int_0^{2\pi} \int_0^R \frac{3\mu v_0 R^2}{[H(t)]^3} \left[1 - \left(\frac{r}{R} \right)^2 \right] r dr d\theta = 2\pi R^2 \frac{3\mu v_0 R^2}{[H(t)]^3} \int_0^1 (1 - \xi^2) \xi d\xi$$

The integral is 1/4, and this leads to the result in Eq. 3C.1-14.

f. In this situation, the radius of the glob of liquid $R(t)$ and the instantaneous disk separation $H(t)$ are related to the sample volume V by $V = \pi[R(t)]^2 H(t)$. Then the force acting on the upper disk is

$$F(t) = \int_0^{2\pi} \int_0^{R(t)} \frac{3\mu v_0 [R(t)]^2}{[H(t)]^3} \left[1 - \left(\frac{r}{R(t)} \right)^2 \right] r dr d\theta$$

$$= \frac{3\pi\mu v_0 [R(t)]^4}{2[H(t)]^3} = \frac{3\mu v_0 V^2}{2\pi[H(t)]^5}$$

g. If, in Eq. 3C.1-14 we replace v_0 by $-dH/dt$, we then have an ordinary, separable differential equation for $H(t)$. Integration gives

$$\frac{2F_0}{3\pi\mu R^4} \int_0^t dt = - \int_{H_0}^H \frac{dH}{H^3}$$

whence

$$\frac{1}{[H(t)]^2} = \frac{1}{H_0^2} + \frac{4F_0 t}{3\pi\mu R^4}$$

3C.2 Normal stresses at solid surfaces for compressible fluids

First write the equation of continuity for a compressible fluid as

$$\frac{\partial}{\partial t} \ln \rho = -(\nabla \cdot \mathbf{v}) - \frac{1}{\rho} (\mathbf{v} \cdot \nabla \rho)$$

The normal stress on a surface perpendicular to the z -axis is

$$\begin{aligned} \tau_{zz}|_{z=0} &= \left(-2\mu \frac{\partial v_z}{\partial z} + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) \right) \Big|_{z=0} \\ &= \left(2\mu \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial}{\partial t} \ln \rho + \frac{1}{\rho} (\mathbf{v} \cdot \nabla \rho) \right) + \left(\frac{2}{3}\mu - \kappa \right) \left(-\frac{\partial}{\partial t} \ln \rho + \frac{1}{\rho} (\mathbf{v} \cdot \nabla \rho) \right) \right) \Big|_{z=0} \\ &= \left(\frac{4}{3}\mu + \kappa \right) \left(\frac{\partial}{\partial t} \ln \rho \right) \Big|_{z=0} \end{aligned}$$

The terms containing \mathbf{v} drop out by the no-slip condition at the surface, and their derivatives with respect to x and y drop out on the surface as explained in Example 3.1-1. This result shows that the normal stresses at surfaces are zero for compressible flow if the flow is at steady state.

3C.3 Deformation of a fluid line

The curve at any time t is $\theta(r, t) = (v_\theta/r)t$, which in tangential annular flow is (from Eq. 3.6-32)

$$\theta(r, t) = \left(\frac{(R/r)^2 - 1}{(1/\kappa)^2 - 1} \right) \Omega_i t \quad \text{and} \quad d\theta = -\frac{2(R^2/r^3)\Omega_i t}{(1/\kappa)^2 - 1} dr$$

The differential element of length along the curve is given by

$$(dl)^2 = (dr)^2 + (rd\theta)^2 = (dr)^2 \left[1 + \frac{4(R/r)^4 (\Omega_i t)^2}{((1/\kappa)^2 - 1)^2} \right]$$

The total length of the curve is then

$$l = \int_0^l d\bar{l} = \int_{\kappa R}^R \sqrt{1 + \frac{4(R/r)^4 (2\pi N)^2}{((1/\kappa)^2 - 1)^2}} dr = R \int_{\kappa}^1 \sqrt{1 + \frac{16\pi^2 N^2}{((1/\kappa)^2 - 1)^2 \xi^4}} d\xi$$

To get a rough, order-of-magnitude estimate assume that N is large and then the "1" can be neglected and the integral performed analytically

$$\frac{l}{R} \approx \frac{4\pi N \kappa}{1 + \kappa} \quad (\text{limit of large } N)$$

3C.4 Alternative methods of solving the Couette viscometer problem by use of angular momentum concepts

a. By making an angular momentum balance (actually the z-component of the angular momentum balance) over an annular region of thickness Δr and height L we obtain

$$(2\pi rL) \cdot (r\tau_{r\theta}) \Big|_r - (2\pi(r+\Delta r)L) \cdot (r\tau_{r\theta}) \Big|_{r+\Delta r} = 0$$

Dividing through by $2\pi L\Delta r$ and rearranging we get

$$\frac{(r^2\tau_{r\theta}) \Big|_{r+\Delta r} - (r^2\tau_{r\theta}) \Big|_r}{\Delta r} = 0$$

whence

$$\frac{d}{dr}(r^2\tau_{r\theta}) = 0$$

the second form resulting from taking the limit as $\Delta r \rightarrow 0$. Then using Eq. B.1-11 for the stress-tensor component, we have

$$\frac{d}{dr}\left(r^3 \frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right) = 0$$

whence

$$v_\theta = -\frac{C_1}{2r} + C_2r$$

From this Eq 3.6-20 follows.

b. Here we start with Eq. 3.4-1, which simplifies to the following for the symmetric stress tensor

$$[\nabla \cdot \{\mathbf{r} \times \boldsymbol{\tau}\}^\dagger] = 0$$

The z-component of this equation is

$$\frac{1}{r} \frac{d}{dr} (r \{ \mathbf{r} \times \boldsymbol{\tau} \}_{rz}^+) = 0 \quad \text{or} \quad \frac{1}{r} \frac{d}{dr} (r \{ \mathbf{r} \times \boldsymbol{\tau} \}_{zr}) = 0$$

where, in cylindrical coordinates, $\mathbf{r} = \boldsymbol{\delta}_r r + \boldsymbol{\delta}_z z$. We now work out the cross product, which is

$$\{ \mathbf{r} \times \boldsymbol{\tau} \}_{zr} = \sum_i \sum_j \varepsilon_{zij} (\boldsymbol{\delta}_r r + \boldsymbol{\delta}_z z)_i \tau_{jr} = \varepsilon_{zr\theta} r \tau_{\theta r} + \varepsilon_{z\theta r} (0) \tau_{rr}$$

Hence the equation of change for angular momentum simplifies to

$$\frac{1}{r} \frac{d}{dr} (r^2 \tau_{r\theta}) = 0$$

and the development proceeds further as in (a).

3C.5 Two-phase interfacial boundary conditions

a. This result follows at once from Eq. 3C.5-1, when the viscous-stress-tensor terms are omitted.

b. To get the right side of Eq. 3C.5-3, it is evident that Eq. 3C.5-1 had to be multiplied by $1/\rho^I v_0^2$. The interfacial-tension term in Eq. 3C.5-3 is then

$$\mathbf{n}^I \left(\frac{l_0}{R_1} + \frac{l_0}{R_2} \right) \left[\left[\frac{\sigma}{l_0 v_0^2 \rho^I} \right] \right] = \mathbf{n}^I \left(\frac{1}{\bar{R}_1} + \frac{1}{\bar{R}_2} \right) \left[\left[\frac{\sigma}{l_0 v_0^2 \rho^I} \right] \right]$$

The terms involving the viscous stress tensor are

$$\frac{[\mathbf{n}^I \cdot \boldsymbol{\tau}^I]}{\rho^I v_0^2} = -\frac{\mu^I [\mathbf{n}^I \cdot \boldsymbol{\gamma}^I]}{\rho^I v_0^2} = -\frac{\mu^I [\mathbf{n}^I \cdot \tilde{\boldsymbol{\gamma}}^I] v_0}{\rho^I v_0^2 l_0} = -[\mathbf{n}^I \cdot \tilde{\boldsymbol{\gamma}}^I] \left[\left[\frac{\mu^I}{l_0 v_0 \rho^I} \right] \right]$$

$$\frac{[\mathbf{n}^I \cdot \boldsymbol{\tau}^{II}]}{\rho^I v_0^2} = -\frac{\mu^{II} [\mathbf{n}^I \cdot \boldsymbol{\gamma}^{II}]}{\rho^I v_0^2} = -\frac{\mu^{II} [\mathbf{n}^I \cdot \tilde{\boldsymbol{\gamma}}^{II}] v_0}{\rho^I v_0^2 l_0} = -[\mathbf{n}^I \cdot \tilde{\boldsymbol{\gamma}}^{II}] \left[\left[\frac{\mu^{II}}{l_0 v_0 \rho^{II}} \right] \right] \left[\left[\frac{\rho^{II}}{\rho^I} \right] \right]$$

And finally, the pressure terms are converted to modified pressure terms plus terms involving the gravitational acceleration

$$\mathbf{n}^I \left(\frac{p^I - p_0 + \rho^I g(h - h_0)}{\rho^I v_0^2} - \frac{p^{II} - p_0 + \rho^{II} g(h - h_0)}{\rho^I v_0^2} \right) - \mathbf{n}^I \frac{g(h - h_0)(\rho^I - \rho^{II})}{\rho^I v_0^2}$$

$$\mathbf{n}^I \left(\mathfrak{P}^I - \mathfrak{P}^{II} \left[\left[\frac{\rho^{II}}{\rho^I} \right] \right] \right) - \mathbf{n}^I \left[\left[\frac{g l_0}{v_0^2} \right] \right] \left[\left[\frac{\rho^I - \rho^{II}}{\rho^I} \right] \right] \tilde{h}$$

We see that the Reynolds numbers for the two phases, the Weber number (Eq. 3.7-12), and the Froude number (Eq. 3.7-11) appear as well as the density ratios for the two phases.

3D.1 Derivation of the equation of motion from Newton's second law of motion

a. Equation 3D.1-1 is the statement that the time rate-of-change of momentum is equal to the sum of the surface forces and the gravity forces acting on a small blob of fluid.

When the Leibniz formula (Eq. A.5-5) is applied to the left side of Eq. 3D.1-1, we get

$$\begin{aligned} \frac{d}{dt} \int_{V(t)} \rho \mathbf{v} dV &= \int_{V(t)} \frac{\partial}{\partial t} \rho \mathbf{v} dV + \int_{S(t)} (\rho \mathbf{v})(\mathbf{v} \cdot \mathbf{n}) dS = \int_{V(t)} \frac{\partial}{\partial t} \rho \mathbf{v} dV + \int_{S(t)} [\mathbf{n} \cdot \rho \mathbf{v} \mathbf{v}] dS \\ &= \int_{V(t)} \frac{\partial}{\partial t} \rho \mathbf{v} dV + \int_{V(t)} [\nabla \cdot \rho \mathbf{v} \mathbf{v}] dV \quad (\text{using Eq. A.5-3}) \end{aligned}$$

The term containing the stress tensor in Eq. 3D.1-1 can also be rewritten as a volume integral using Eq. A.5-3 to give

$$\int_{V(t)} \frac{\partial}{\partial t} \rho \mathbf{v} dV = - \int_{V(t)} [\nabla \cdot \rho \mathbf{v} \mathbf{v}] dV - \int_{V(t)} [\nabla \cdot \boldsymbol{\pi}] dV + \int_{V(t)} \rho \mathbf{g} dV$$

Since the choice of the blob volume was arbitrary, all the volume integral operations may be removed, and we obtain the equation of motion of Eq. 3.2-9.

b. If the blob is fixed, then we can write a momentum balance over the blob as follows:

$$\frac{d}{dt} \int_V \rho \mathbf{v} dV = - \int_S [\mathbf{n} \cdot \rho \mathbf{v} \mathbf{v}] dS - \int_S [\mathbf{n} \cdot \boldsymbol{\pi}] dS + \int_V \rho \mathbf{g} dV$$

This states that the rate of increase of momentum within the fixed volume equals the rate of increase because of convective transport, the rate of increase because of molecular momentum transport, and the force acting on the system by gravity. The time derivative can be taken inside, since the volume is fixed, and the surface integrals can be converted to volume integrals. The result is an equation containing only volume integrals over the fixed volume:

$$\int_V \frac{\partial}{\partial t} \rho \mathbf{v} dV = - \int_V [\nabla \cdot \rho \mathbf{v} \mathbf{v}] dV - \int_V [\nabla \cdot \boldsymbol{\pi}] dV + \int_V \rho \mathbf{g} dV$$

Since the volume was chosen arbitrarily, the volume integrals can be removed, and, once again, the result in Eq. 3.2-9 is obtained.

3D.2 The equation of change for vorticity

Method I:

Start with the Navier-Stokes equation in the D/Dt -form, but rearranged thus:

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial t} &= -[\mathbf{v} \cdot \nabla \mathbf{v}] - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{g} \\ &= -\nabla \frac{1}{2} v^2 + [\mathbf{v} \times [\nabla \times \mathbf{v}]] - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{g}\end{aligned}$$

Next we take the curl and introduce the vorticity $\mathbf{w} = [\nabla \times \mathbf{v}]$

$$\frac{\partial \mathbf{w}}{\partial t} = [\nabla \times [\mathbf{v} \times \mathbf{w}]] + \nu \nabla^2 \mathbf{w}$$

or

$$\frac{\partial \mathbf{w}}{\partial t} = [\nabla \cdot \mathbf{w} \mathbf{v}] - [\nabla \cdot \mathbf{v} \mathbf{w}] + \nu \nabla^2 \mathbf{w}$$

Then using Eq. A.4-24 and the fact that $(\nabla \cdot \mathbf{v}) = 0$ for incompressible fluids and $(\nabla \cdot \mathbf{w}) = 0$ always (since the divergence of a curl is always zero, we get Eq. 3D.2-1.

Method II:

Start now with the Navier-Stokes equation in $\partial/\partial t$ -form

$$\frac{\partial \mathbf{v}}{\partial t} = -[\nabla \cdot \mathbf{v} \mathbf{v}] - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{g}$$

Take the curl of this equation and introduce the vorticity to get

$$\frac{\partial \mathbf{w}}{\partial t} = -[\nabla \times [\nabla \cdot \mathbf{v} \mathbf{v}]] + \nu \nabla^2 \mathbf{w}$$

or

$$\frac{\partial \mathbf{w}}{\partial t} = [\boldsymbol{\varepsilon} : \{\nabla \mathbf{v} \cdot \nabla \mathbf{v}\}] - [\mathbf{v} \cdot \nabla \mathbf{w}] + \nu \nabla^2 \mathbf{w}$$

Details of the manipulations involved in this last step are given here using the abbreviated notation of §A.9 with the Einstein summation convention:

$$\begin{aligned}
 -[\nabla \times [\nabla \cdot \mathbf{v}\mathbf{v}]]_i &= -\varepsilon_{ijk} \partial_j (\partial_l v_l v_k) = -\varepsilon_{ijk} \partial_j (v_l \partial_l v_k - v_k \partial_l v_l) \text{ but } \partial_l v_l = 0 \\
 &= -\varepsilon_{ijk} \left[(\partial_j v_l) (\partial_l v_k) + (v_l \partial_j \partial_l v_k) \right] \\
 &= \varepsilon_{ikj} (\partial_j v_l) (\partial_l v_k) - (v_l \partial_l (\varepsilon_{ijk} \partial_j v_k)) \\
 &= [\boldsymbol{\varepsilon} : \{\nabla \mathbf{v} \cdot \nabla \mathbf{v}\}]_i - [\mathbf{v} \cdot \nabla [\nabla \times \mathbf{v}]]_i
 \end{aligned}$$

in which $\boldsymbol{\varepsilon} = \Sigma \Sigma \Sigma \varepsilon_{mnr} \boldsymbol{\delta}_m \boldsymbol{\delta}_n \boldsymbol{\delta}_r$ is a third-order tensor.

3D.3 Alternate form of the equation of motion

Take the divergence of the equation of motion for an incompressible fluid in the form of Eq. 3.2-9, but with the stress-tensor term written in terms of the viscosity and the Laplacian of the velocity. This gives

$$0 = -(\nabla \cdot [\nabla \cdot \mathbf{v}\mathbf{v}]) - \frac{1}{\rho} \nabla^2 p$$

or

$$0 = -(\nabla \mathbf{v} : (\nabla \mathbf{v})^{\dagger}) - \frac{1}{\rho} \nabla^2 p$$

Then use the definitions in Eq. 3D.3-2 to get Eq. 3D.3-1.

4A.1 Time for attainment of steady state in tube flow.

a. In Figure 4D.2, the centerline velocity comes within 10% of its final value when $\nu t/R^2 = 0.45$, giving

$$t = 0.45R^2/\nu = (0.45)(0.49 \times 10^{-4} \text{ m}^2)/(3.45 \times 10^{-4} \text{ m}^2/\text{s}) = 0.064 \text{ s}$$

b. If water at (68°F=20°C) is used, with $\nu = 1.0037 \times 10^{-6} \text{ m}^2/\text{s}$ from Tables 1.1-1 and F.3-6, the time required is

$$t = (0.45)(0.49 \times 10^{-4} \text{ m}^2)/(1.0037 \times 10^{-6} \text{ m}^2/\text{s}) = 22 \text{ s}$$

4A.2 Velocity near a moving sphere.

From Eq. 4.2-14 at $\theta = \pi/2$, the fluid velocity relative to the approach velocity falls to 1% of v_∞ at $v_\theta = -0.99v_\infty$ relative to the sphere, giving

$$-0.99 = - \left[1 - \frac{3}{4} \left(\frac{R}{r} \right) - \frac{1}{4} \left(\frac{R}{r} \right)^3 \right]$$

with $R/r < 1$. If $R/r \ll 1$, the cubic term will be unimportant, giving

$$0.01 = \frac{3}{4} \left(\frac{R}{r} \right)$$

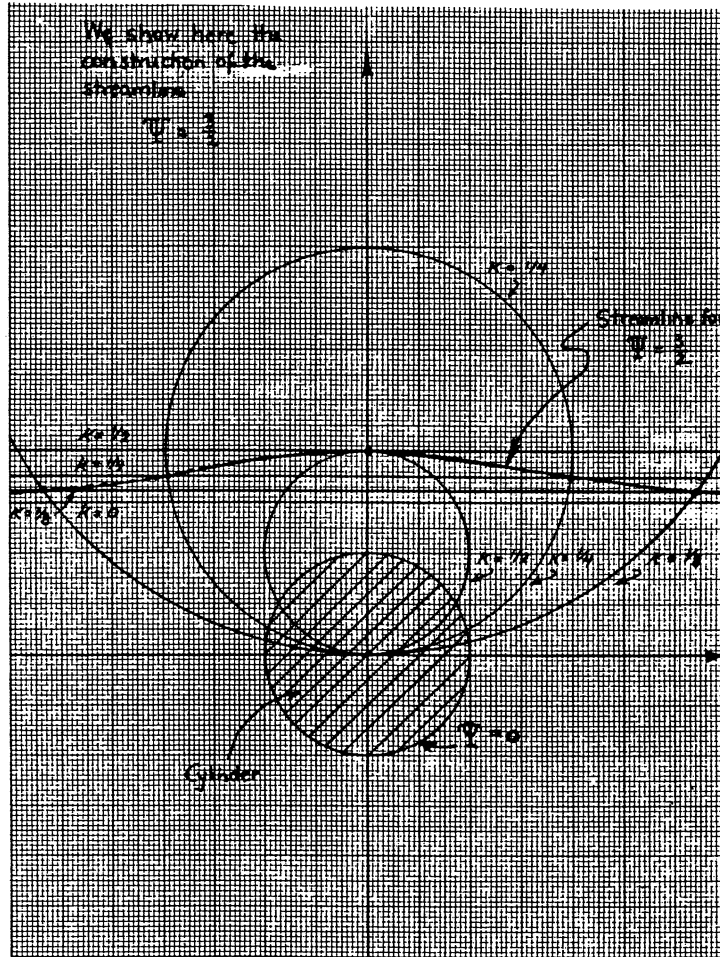
or

$$\frac{r}{R} = \frac{300}{4} = 75$$

Clearly, the neglect of the cubic term at this distance is justifiable.

4A.3 Construction of streamlines for the potential flow around a cylinder.

In the following drawing we show the construction of the streamline $\Psi = \frac{3}{2}$ by the method described in the problem statement.



4A.4 Comparison of exact and approximate profiles for flow along a flat plate.

Let $\Pi_v = v_x/v_\infty$ and $Y = y\sqrt{v_\infty/\nu x}$; then Eq. 4.4-18 gives the approximation

$$\begin{aligned}\Pi_v &= \frac{3}{2}\sqrt{\frac{13}{280}}Y - \frac{1}{2}\left(\frac{13}{280}\right)^{3/2}Y^3 \\ &= 0.323209Y - 0.005002Y^3\end{aligned}$$

and Fig. 4.4-3 shows Blasius' "exact" Π_v vs. Y . These two velocity representations will now be compared.

Location, Y	Approx. Π_v , Eq. 4.4-18	"Exact" Π_v , Fig. 4.4-3	Approx. Π_v /Exact Π_v
a. 1.5	0.468	0.49	0.96
b. 3.0	0.835	0.84	0.99
c. 4.0	0.973	0.96	1.01

4A.5 Numerical demonstration of the von Karman momentum balance.

a. The integrals in Eq. 4.4-13 are

$$I_1 = \int_0^{\infty} \rho v_x (v_e - v_x) dy \quad \text{and} \quad I_2 = \int_0^{\infty} \rho (v_e - v_x) dy$$

Figure 4.4-3 gives $f' = v_x/v_{\infty}$ as a function of a dimensionless coordinate,

$$Y = y\sqrt{v_{\infty}/(\nu x)}$$

and $v_e = v_{\infty}$ in this geometry. Thus, $v_x = v_{\infty}f'$ and $\rho dy = \sqrt{\rho\mu x/v_{\infty}}dY$, so that these integrals take the forms

$$I_1 = \sqrt{\rho\mu v_{\infty}x} \int_0^{\infty} f'(1-f')dY \quad \text{and} \quad I_2 = \sqrt{\rho\mu v_{\infty}x} \int_0^{\infty} (1-f')dY$$

Numerical evaluation of the integrals over Y then gives

$$\int_0^{\infty} f'(1-f')dY = 0.664 \quad \text{and} \quad \int_0^{\infty} (1-f')dY = 1.73$$

if an accurate table of the solution is used. The following calculation was made from Fig. 4.4-3:

Y	0	1.0	2.0	3.0	4.0	5.0	6.0
f'	0	0.34	0.63	0.844	0.955	0.983	1.00
$(1-f')$	1.	0.66	0.37	0.156	0.045	0.017	0.000
$f'(1-f')$	0	0.2244	0.2331	0.1309	0.0430	0.0167	0.000

Application of the trapezoidal rule gives the values

$$I_1 = [1/2 + 0.66 + 0.37 + 0.156 + 0.045 + 0.017 + 0/2] \times 1.0 = 1.74$$

$$I_2 = [0/2 + 0.2244 + 0.2331 + 0.1309 + 0.043 + 0.0167 + 0/2] \times 1.0 = 0.65$$

which agree, within their uncertainty, with the accepted values 1.73 and 0.664.

b. Use of Eq. 1.1-2 and the results of *a.* in Eq. 4.4-18 gives

$$\begin{aligned} \tau_{yx}|_{y=0} &= \frac{dI_1}{dx} + (0)I_2 \\ &= 0.65\sqrt{\rho\mu v_{\infty}^3} \frac{x^{-1/2}}{2} \\ &= 0.325\sqrt{\rho\mu v_{\infty}^3/x} \end{aligned}$$

c. The force in the x-direction on a plate of width W and length L , wetted on both sides, according to the result in *b.* is

$$\begin{aligned} F_x &= 2W \int_0^L \tau_{yx}|_{y=0} dx \\ &= 2W(0.325\sqrt{\rho\mu v_{\infty}^3}) \int_0^L x^{-1/2} dx \\ &= 1.30\sqrt{\rho\mu v_{\infty}^3} LW^2; \quad \text{The recommended coefficient is 1.328.} \end{aligned}$$

4A.6 Use of boundary-layer formulas.

The data for this problem are:

$$W = 10 \text{ ft}$$

$$L = 3 \text{ ft}$$

$$v_{\infty} = 20 \text{ ft/s}$$

From Table 1.1-2 and Appendix F:

$$\begin{aligned}\nu &= (0.1505 \text{ cm}^2/\text{s}) / (12 \times 2.54 \text{ cm/ft})^2 \\ &= 1.62 \times 10^{-4} \text{ ft}^2/\text{s}\end{aligned}$$

$$\begin{aligned}\mu &= (0.01813 \text{ mPa}\cdot\text{s})(10^{-3} \text{ Pa/mPa})(6.7197 \times 10^{-1} \text{ lb}_m/\text{ft}\cdot\text{s}/\text{Pa}) \\ &= 1.218 \times 10^{-5} \text{ lb}_m/\text{ft}\cdot\text{s}\end{aligned}$$

$$\rho = \mu/\nu = 1.218 \times 10^{-5} / 1.62 \times 10^{-4} = 7.5 \times 10^{-2} \text{ lb}_m/\text{ft}^3$$

a. The local Reynolds number at the trailing edge ($x = L = 20 \text{ ft}$) is:

$$\begin{aligned}\text{Re} &= Lv_{\infty}/\nu \\ &= (3\text{ft})(20 \text{ ft/s}) / (1.62 \times 10^{-4} \text{ ft}^2/\text{s}) \\ &= 3.7 \times 10^5\end{aligned}$$

b. According to Eq. 4.4-17, the boundary layer thickness at the trailing edge is

$$\begin{aligned}\delta(L) &= 4.64 \sqrt{\frac{\nu L}{v_{\infty}}} \\ &= 4.64 \sqrt{\frac{(1.62 \times 10^{-4} \text{ ft}^2/\text{s})(3 \text{ ft})}{20 \text{ ft/s}}} \\ &= 2.4 \times 10^{-5} \text{ ft}\end{aligned}$$

c. According to Eq. 4.4-30, the total drag force of the fluid on both sides of the plate is

$$\begin{aligned}F_x &= 1.328 \sqrt{\rho \mu L W^2 v_{\infty}^3} \\ &= 1.328 \sqrt{(0.075 \text{ lb}_m/\text{ft}^3)(1.22 \times 10^{-5} \text{ lb}_m/\text{ft}\cdot\text{s})(3 \text{ ft})(10 \text{ ft})^2(20 \text{ ft/s})^3} \\ &= 0.62 \text{ lb}_m\cdot\text{ft}/\text{s}^2 \\ &= 0.019 \text{ lb}_f\end{aligned}$$

4A.7 Entrance flow in conduits.

a. With the indicated substitutions, Eq. 4.4-17 gives

$$\frac{1}{2}D = 4.64\sqrt{\frac{\nu L_e}{v_{\max}}}$$

Setting $v_{\max} = 2\langle v \rangle$ at the end of the entrance region, we obtain the following estimate of L_e :

$$\frac{\nu L_e}{2\langle v \rangle} = \left(\frac{D}{9.28}\right)^2$$

or

$$\begin{aligned} L_e &= \frac{2}{(9.28)^2} \frac{D^2 \langle v \rangle}{\nu} \\ &= 0.023DRe \end{aligned}$$

which is similar to the expression given in §2.3, except that the coefficient is about 2/3 as large.

b. At the typical transition locus $xv_{\infty}/\nu \approx 3 \times 10^5$ for flow along a flat plate, Eq. 4.4-17 gives

$$\begin{aligned} \delta/x &= 4.64\sqrt{\frac{\nu}{v_{\infty}x}} \\ &= 4.64(3.5 \times 10^5)^{-1/2} = 0.00847 \end{aligned}$$

and the transition Reynolds number based on the characteristic length δ is

$$\begin{aligned} \frac{v_{\infty}\delta}{\nu} &= \frac{v_{\infty}x}{\nu} \frac{\delta}{x} \\ &= (3 \times 10^5)(0.00847) = 2542 \end{aligned}$$

For flow in tubes, with transition occurring when $\delta = D/2$ and with $v_{\infty} = 2\langle v \rangle$, the latter result gives $D\langle v \rangle/\nu = 2542$ as the minimum transition Reynolds number, in fair agreement with the reported value of 2100.

c. For laminar flow between parallel planes, the method in Problem 4.C gives $\delta = B$ and $v_{\max} = (3/2)\langle v_x \rangle$ at the end of the entrance region. Insertion of these results into Eq. 4.4-17 gives

$$B = 4.64\sqrt{\frac{\nu L_e}{1.5\langle v_x \rangle}}$$

whence

$$\begin{aligned} L_e &= \frac{1.5}{(4.64)^2} \frac{B^2 \langle v_x \rangle}{\nu} \\ &= 0.070B^2 \langle v_x \rangle / \nu \\ &= 0.070BRe \text{ with } Re = B\langle v_x \rangle / \nu \end{aligned}$$

4B.1 Flow of a fluid with a suddenly applied constant wall stress

a. Differentiation of Eq. 4.1-1 with respect to y gives

$$\frac{\partial^2 v_x}{\partial y \partial t} = \nu \frac{\partial}{\partial y} \frac{\partial^2 v_x}{\partial y^2} \quad \text{or} \quad \frac{\partial}{\partial t} \left(-\mu \frac{\partial v_x}{\partial y} \right) = \nu \frac{\partial^2}{\partial y^2} \left(-\mu \frac{\partial v_x}{\partial y} \right)$$

Then using Newton's law of viscosity, we get

$$\frac{\partial \tau_{yx}}{\partial t} = \nu \frac{\partial^2 \tau_{yx}}{\partial y^2}$$

b. This equation is to be solved with the initial condition that $\tau_{yx} = 0$ for $t \leq 0$, and the boundary conditions that $\tau_{yx} = \tau_0$ at $y = 0$, and that $\tau_{yx} = 0$ at $t = \infty$.

c. The solution is exactly as in Example 4.1-1 with appropriate changes of notation, and the solution is given in Eq. 4B.1-1.

d. To get the velocity profile, we integrate Newton's law of viscosity:

$$\int_{v_x}^0 d\bar{v}_x = -\frac{1}{\mu} \int_y^\infty \tau_{yx} dy \quad \text{or} \quad v_x = \frac{\tau_0}{\mu} \int_y^\infty \left(1 - \operatorname{erf} \frac{\bar{y}}{\sqrt{4\nu t}} \right) d\bar{y}$$

Changing variables we get

$$\begin{aligned} v_x &= \frac{\tau_0}{\mu} \sqrt{4\nu t} \int_{y/\sqrt{4\nu t}}^\infty (1 - \operatorname{erf} u) du \\ &= \frac{\tau_0}{\mu} \sqrt{4\nu t} \left[\frac{1}{\sqrt{\pi}} e^{-y^2/4\nu t} - \frac{\bar{y}}{\sqrt{4\nu t}} \left(1 - \operatorname{erf} \frac{y}{\sqrt{4\nu t}} \right) \right] \end{aligned}$$

The velocity at $y = 0$ is then

$$v_x(0, t) = \frac{\tau_0}{\mu} \sqrt{\frac{4\nu t}{\pi}} = 2\tau_0 \sqrt{\frac{t}{\pi\mu\rho}}$$

4B.2 Flow near a wall suddenly set in motion (approximate solution)

a. Integration of Eq. 4.1-1 over y gives

$$\rho \int_0^{\infty} \frac{\partial v_x}{\partial t} dz = \mu \int_0^{\infty} \frac{\partial^2 v_x}{\partial y^2} dz \quad \text{whence} \quad \frac{d}{dt} \int_0^{\infty} \rho v_x dz = \mu \left. \frac{\partial v_x}{\partial y} \right|_{y=0}^{y=\infty}$$

Since the velocity gradient is zero at infinite distance from the plate, we end up with Eq. 4B.2-2.

b. We introduce the variable $\eta = y/\delta(t)$. Then when Eqs. 4B.2-3 and 4 are substituted in Eq. 4B.2-2, we find

$$\frac{d}{dt} \delta(t) \int_0^1 \rho v_{\infty} \left(1 - \frac{3}{2} \eta + \frac{1}{2} \eta^3\right) d\eta = -\mu v_{\infty} \left(-\frac{3}{2} + \frac{3}{2} \eta^2\right) \Big|_{\eta=0} \frac{1}{\delta(t)}$$

Then after dividing by ρv_{∞} and evaluating of the integral, we get

$$\frac{3}{8} \frac{d}{dt} \delta(t) = \frac{3}{2} \nu \frac{1}{\delta(t)} \quad (\text{see Eq. 4B.2-5})$$

c. Eq. 4B.2-5 when integrated gives

$$\int_0^{\delta} \delta d\delta = 4\nu \int_0^t dt \quad \text{or} \quad \delta(t) = \sqrt{8\nu t}$$

Then the velocity profile is given by

$$\frac{v_x}{v_{\infty}} = 1 - \frac{3}{2} \left(\frac{y}{\sqrt{8\nu t}} \right) + \frac{1}{2} \left(\frac{y}{\sqrt{8\nu t}} \right)^3 \quad \text{for} \quad 0 \leq y \leq \sqrt{8\nu t}$$

$$\frac{v_x}{v_{\infty}} = 1 \quad \text{for} \quad y \geq \sqrt{8\nu t}$$

4B.3 Creeping flow around a spherical bubble

a. According to Eq. B.1-18, the vanishing of the shear stress is

$$r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} = 0 \quad \text{or} \quad r \frac{\partial}{\partial r} \left(\frac{1}{r} \left\{ \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \right\} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left\{ -\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \right\} = 0$$

In the second form, we have inserted the expressions for the velocity components in terms of the stream function from the last line in Table 4.2-1. Next we insert $\psi = f(r) \sin^2 \theta$ and obtain Eq. 4B.3-1.

b. Equations 4.2-7 through 10 are still valid for this problem, as well as the values of $C_3 = -\frac{1}{2}v_\infty$ and $C_4 = 0$. Hence we have to determine the remaining constants in $f(r) = C_1 r^{-1} + C_2 r$ by requiring that Eq. 4B.3-1 be valid at $r = R$, as well as $f = 0$ at $r = R$ (Eq. 4.2-3). These boundary conditions lead to $C_1 = 0$ and $C_2 = \frac{1}{2}v_\infty R$. Then Eqs. 4B.3-2 and 3 follow directly.

c. When the velocity distributions in Eq. 4B.3-2 and 3 are substituted into the equations of motion (Eqs. B.6-7 and 8), we get

$$\frac{\partial \mathcal{P}}{\partial r} = 2 \left(\frac{\mu v_\infty}{R^2} \right) \left(\frac{R}{r} \right)^3 \cos \theta \quad \text{and} \quad \frac{\partial \mathcal{P}}{\partial \theta} = \left(\frac{\mu v_\infty}{R} \right) \left(\frac{R}{r} \right)^2 \sin \theta$$

Integration of these two equations gives

$$\mathcal{P} = - \left(\frac{\mu v_\infty}{R} \right) \left(\frac{R}{r} \right)^2 \cos \theta + F(\theta) \quad \text{and} \quad \mathcal{P} = - \left(\frac{\mu v_\infty}{R} \right) \left(\frac{R}{r} \right)^2 \cos \theta + G(r)$$

In order for the solution to be unique, $F(\theta)$ and $G(r)$ must be equal to a constant. If we require that the modified pressure be equal to p_0 at $z = 0$ infinitely far from the sphere, we then get

$$p = p_0 - \rho g z - \left(\frac{\mu v_\infty}{R} \right) \left(\frac{R}{r} \right)^2 \cos \theta$$

d. The z-component of the force acting on the sphere is

$$F_z = - \int_0^{2\pi} \int_0^\pi \left(\delta_z \cdot \left[\delta_r \cdot \{ p \delta + \tau \} \right] \right) \Big|_{r=R} R^2 \sin \theta d\theta d\phi$$

$$= -2\pi R^2 \int_0^\pi (p \cos \theta + \tau_{rr} \cos \theta - \tau_{r\theta} \sin \theta) \Big|_{r=R} \sin \theta d\theta$$

where Eqs. A.6-28 and 29 have been used for getting the dot products of the unit vectors. Then we use the result of (c) and Eqs. B.1-15 and 18 for the components of the stress tensor (along with Eqs. 4B.3-2 and 3) to get the three contributions to the force:

$$\begin{aligned} F_{z,p} &= -2\pi R^2 \int_0^\pi \left(p_0 - \rho g z - \left(\frac{\mu v_\infty}{R} \right) \left(\frac{R}{r} \right)^2 \cos \theta \right) \Big|_{r=R} \cos \theta \sin \theta d\theta \\ &= \frac{4}{3} \pi R^3 \rho g + \frac{4}{3} \pi \mu R v_\infty \end{aligned}$$

$$\begin{aligned} F_{z,\tau_{rr}} &= -2\pi R^2 \int_0^\pi \left(-2\mu \frac{\partial v_r}{\partial r} \cdot \cos \theta \right) \Big|_{r=R} \sin \theta d\theta \\ &= +4\pi \mu R^2 v_\infty \int_0^\pi \left(\frac{R}{r^2} \cdot \cos^2 \theta \right) \Big|_{r=R} \sin \theta d\theta = \frac{8}{3} \pi \mu R v_\infty \end{aligned}$$

$$F_{z,\tau_{r\theta}} = -2\pi R^2 \int_0^\pi \left(-\mu \left(r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \right) \right) \Big|_{r=R} (-\sin \theta) \sin \theta d\theta = 0$$

When these are added together, we get Eq. 4B.3-5.

4B.4 Use of the vorticity equation

a. For the postulate $v_z = v_z(x)$ the vorticity $\mathbf{w} = [\nabla \times \mathbf{v}]$ has but one component $w_y = -dv_z/dx$. Then at steady state, the y -component of Eq. 3D.2-1 is

$$(\mathbf{v} \cdot \nabla) \left(\frac{dv_z}{dx} \right) = v \nabla^2 \left(\frac{dv_z}{dx} \right) + (\mathbf{w} \cdot \nabla) v_y$$

But v_y is postulated to be zero so that the last term drops out. Also the first term drops out, because v_x and v_y are zero, and v_z is postulated to have no z -dependence. Consequently the equation simplifies to $d^3v_z/dx^3 = 0$. Integration gives

$$v_z = \frac{1}{2}C_1x^2 + C_2x + C_3 \quad \text{or} \quad v_z/v_{z,\max} = D_1\xi^2 + D_2\xi + D_3$$

where we have redefined the integration constants. From the three boundary conditions it is found that $D_1 = -1$, $D_2 = 0$, and $D_3 = 1$, Eq. 4B.4-1 results.

Then the z -component of the equation of motion becomes at steady state

$$\rho(\mathbf{v} \cdot \nabla)v_z = -\frac{d\mathcal{P}}{dz} + \mu\nabla^2v_z \quad \text{or} \quad 0 = -\frac{d\mathcal{P}}{dz} + \mu\frac{d^2v_z}{dx^2}$$

Knowing the velocity distribution we can evaluate the second derivative of the velocity and get the pressure distribution thus

$$0 = -\frac{d\mathcal{P}}{dz} + \mu v_{z,\max} \left(-\frac{2}{B^2} \right) \quad \text{and} \quad \mathcal{P}_0 - \mathcal{P}(z) = \frac{2\mu v_{z,\max}}{B^2} z$$

For $z = L$, this gives $v_{z,\max} = (\mathcal{P}_0 - \mathcal{P}_L)B^2/2\mu L$ in agreement with Eq. 2B.3-2.

b. Since it is postulated that $v_z = v_z(r)$, the only nonzero component of the vorticity vector is $w_\theta = -dv_z/dr$. Then the θ -component of the steady-state vorticity equation is:

$$\begin{aligned} \left(v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \right) w_\theta = \\ v \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r w_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 w_\theta}{\partial \theta^2} + \frac{\partial^2 w_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial w_r}{\partial \theta} \right) \\ + \left(w_r \frac{\partial v_\theta}{\partial r} + w_\theta \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) + w_z \frac{\partial v_\theta}{\partial z} \right) \end{aligned}$$

This simplifies to

$$\frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) \right) = 0 \quad \text{or} \quad v_z = \frac{1}{4} C_1 r^2 + C_2 \ln r + C_3$$

This is the same solution found in the solution to Problem 3B.6. When the two boundary conditions and the conservation-of-mass condition are used, we finally get the solution in Eq. 3B.6-2.

To get the pressure distribution we use the z-component of the equation of motion, which is

$$0 = -\frac{d\mathcal{P}}{dr} + \mu \frac{1}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right)$$

When the velocity distribution of Eq. 3B.6-2 is inserted, we get

$$0 = -\frac{d\mathcal{P}}{dr} - \frac{4\mu v_0}{R^2(1-\kappa^2)} \frac{(1-\kappa^2) + 2\kappa^2 \ln(1/\kappa)}{(1-\kappa^2) - 2(1+\kappa^2) \ln(1/\kappa)}$$

This may then be integrated to get the pressure distribution.

4B.5 Steady potential flow around a stationary sphere

a. The boundary conditions are:

(i) as $r \rightarrow \infty$, $\mathbf{v} \rightarrow v_\infty \delta_z$, or by using Eqs. A.6-28 and 29

as $r \rightarrow \infty$, $v_r = v_\infty \cos \theta$ and $v_\theta = -v_\infty \sin \theta$

(ii) at $r = R$, $v_r = 0$

(iii) at $z = 0$ as $r \rightarrow \infty$, $\mathcal{P} \rightarrow \mathcal{P}_0$

b. Since $\mathbf{v} = -\nabla\phi$, we have as $r \rightarrow \infty$

$$-\partial\phi/\partial r = v_\infty \cos \theta \quad \text{and} \quad -(1/r)\partial\phi/\partial\theta = -v_\infty \sin \theta$$

When these equations are integrated we find that $\phi = -v_\infty r \cos \theta$,

Thus we feel that $\phi = f(r)\cos \theta$ may be an appropriate trial function.

c. We next write the 3-dimensional Laplace equation in spherical coordinates (for a system with symmetry about the z-axis)

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) = 0$$

Into this equation we substitute the trial velocity potential and get

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{df}{dr} \right) - 2 \frac{f}{r^2} = 0 \quad \text{which as the solution} \quad f = C_1 r + C_2 r^{-2}$$

since this is an "equidimensional" equation of the form of Eq. C.1-14.

d. Application of the boundary conditions then gives

$$f = -v_\infty R \left[\frac{r}{R} + \frac{1}{2} \left(\frac{R}{r} \right)^2 \right] \quad \text{and} \quad \phi = -v_\infty R \left[\frac{r}{R} + \frac{1}{2} \left(\frac{R}{r} \right)^2 \right] \cos \theta$$

e. Then from the components of $\mathbf{v} = -\nabla\phi$, we get Eqs. 4B.5-2 and 3 by differentiation.

f. Then from the equation of motion for steady potential flow

$$\rho \left(\nabla \frac{1}{2} v^2 \right) = -\nabla \mathcal{P} \quad (\text{see Eq. 4.3-2})$$

By integrating the components of this equation we get

$$-\mathcal{P} = \rho v_{\infty}^2 \left[\left[1 - \left(\frac{R}{r} \right)^3 \right] \cos^2 \theta + \left[1 + \frac{1}{2} \left(\frac{R}{r} \right)^3 \right] \sin^2 \theta \right] + C_3$$

The integration constant is then obtained from the boundary condition given in (a): $C_3 = -\mathcal{P}_0 - \rho v_{\infty}^2$. Then when the modified pressure is evaluated at the surface of the sphere, Eq. 4B.5-4 is obtained directly.

4B.6 Potential flow near a stagnation point

a. At the origin of coordinates ($z = 0$) the complex velocity $dw/dz = -2v_0z$ is zero, which is a stagnation point.

b. By taking the real and imaginary parts of the complex velocity, we get from Eq. 4.3-12: $v_x = 2v_0x$ and $v_y = -2v_0y$.

c. When v_0 is positive, the fluid is flowing toward the surface $y = 0$ in the upper half plane. The magnitude of v_0 specifies the speed with which the fluid is flowing: $v = \sqrt{v_x^2 + v_y^2} = 2v_0r$.

4B.7 Vortex flow

a. By using Eq. 4.3-12, we find that

$$\frac{dw}{dz} = \frac{i\Gamma}{2\pi} \left(\frac{1}{z} \right) = \frac{i\Gamma}{2\pi} \left(\frac{\bar{z}}{z\bar{z}} \right) = \frac{i\Gamma}{2\pi} \left(\frac{x-iy}{x^2+y^2} \right) = -v_x + iv_y$$

in which the overbar indicates the complex conjugate. Equating the real and imaginary parts gives

$$v_x = -\frac{\Gamma}{2\pi} \left(\frac{y}{x^2+y^2} \right) = -\frac{\Gamma}{2\pi} \left(\frac{\sin \theta}{r} \right); \quad v_y = \frac{\Gamma}{2\pi} \left(\frac{x}{x^2+y^2} \right) = \frac{\Gamma}{2\pi} \left(\frac{\cos \theta}{r} \right)$$

The components in cylindrical coordinates are

$$v_r = v_x \cos \theta + v_y \sin \theta = 0; \quad v_\theta = -v_x \sin \theta + v_y \cos \theta = \frac{\Gamma}{2\pi} \left(\frac{1}{r} \right)$$

b. The forced vortex is given by $v_\theta = \Omega r$ in Eq. 3.6-37.

4B.8 The flow field around a line source

a. For this purely radial flow $v_r = v_r(r)$ and the other two components are zero. Then $\nabla^2\phi = 0$ simplifies at once to Eq. 4B.8-1.

b. Integration of Eq. 4B.8-1 gives $\phi = C_1 \ln r + C_2$. Then, since $v_r = -d\phi/dr$, we find $v_r = -C_1/r$. Next we calculate the volumetric flow rate per unit length thus

$$\Gamma = \int_0^{2\pi} v_r r d\theta = -2\pi C_1 \quad \text{whence} \quad v_r = \frac{\Gamma}{2\pi r}$$

The pressure distribution is then obtained from the radial component of the equation of motion

$$\rho v_r \frac{dv_r}{dr} = -\frac{d\mathcal{P}}{dr} \quad \text{or} \quad \rho \left(\frac{\Gamma}{2\pi r} \right) \left(-\frac{\Gamma}{2\pi r^2} \right) = -\frac{d\mathcal{P}}{dr}$$

Integration then gives

$$\int_{\mathcal{P}}^{\mathcal{P}_\infty} d\mathcal{P} = \rho \left(\frac{\Gamma}{2\pi} \right)^2 \int_r^\infty \left(-\frac{1}{r^3} \right) dr \quad \text{or} \quad \mathcal{P}_\infty - \mathcal{P} = \frac{\rho}{2} \left(\frac{\Gamma}{2\pi r} \right)^2$$

4B.9 Checking solutions to unsteady flow problems

a. Substituting the solution of Eq. 4.1-14 (or 15) into Eq. 4.1-5, we have to verify that

$$\frac{\partial}{\partial t} \left(1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\bar{\eta}^2} d\bar{\eta} \right) = \nu \frac{\partial^2}{\partial y^2} \left(1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\bar{\eta}^2} d\bar{\eta} \right)$$

We have to use the chain rule of partial differentiation along with the Leibniz formula

$$\frac{d}{d\eta} \left(\int_0^\eta e^{-\bar{\eta}^2} d\bar{\eta} \right) \cdot \frac{\partial \eta}{\partial t} = \nu \frac{d^2}{d\eta^2} \left(\int_0^\eta e^{-\bar{\eta}^2} d\bar{\eta} \right) \cdot \left(\frac{\partial \eta}{\partial y} \right)^2$$

or

$$\left(e^{-\eta^2} \right) \left(-\frac{1}{2t} \frac{y}{\sqrt{4\nu t}} \right) = \nu \left(e^{-\eta^2} \cdot (-2\eta) \right) \left(\frac{1}{\sqrt{4\nu t}} \right)^2$$

When the definition of η is used, the above is found to be an identity.

4C.1 Laminar entrance flow in a duct.

a. Calculation of the mass flow rate according to Eqs. 4C.1-1 and 4C.1-2 gives

$$\begin{aligned} w &= \rho \langle v_x \rangle W B \\ &= \rho v_e W \int_0^\delta \left[2 \left(\frac{y}{\delta} \right) - \left(\frac{y}{\delta} \right)^2 \right] dy + \rho v_e W \int_\delta^B [1] dy \\ &= \rho v_e W \left[\delta - \frac{\delta}{3} \right] + \rho v_e W [B - \delta] = \rho v_e W \left[-\frac{\delta}{3} + B \right] \end{aligned}$$

Equating the first and last expressions for w , we get (with $\delta < B$ and $\Delta = \delta/B$),

$$v_e(x) = \langle v_x \rangle \frac{B}{B - \delta/3} = \langle v_x \rangle \frac{3}{3 - \Delta}$$

The following related equations will prove useful in part b:

$$\begin{aligned} \frac{dv_e}{dx} &= \langle v_x \rangle \frac{3}{(3 - \Delta)^2} \frac{d\Delta}{dx} \\ v_e \delta \frac{dv_e}{dx} &= \langle v_x \rangle^2 \frac{9B\Delta}{(3 - \Delta)^3} \frac{d\Delta}{dx} \\ \frac{d}{dx} (v_e^2 \delta) &= \frac{d}{dx} \left[\left(\langle v_x \rangle \frac{3}{3 - \Delta} \right)^2 B\Delta \right] \\ &= B \langle v_x \rangle^2 \left[\frac{27 + 9\Delta}{(3 - \Delta)^3} \right] \frac{d\Delta}{dx} \end{aligned}$$

^ob. The boundary layer in this system lies between $y = 0$ and $y = \delta$, so those limits suffice for the integrals in Eq. 4.4-13. Evaluation of the terms in Eq. 4.4-13 (divided by ρ) according to the results in *a* then gives:

$$\nu \frac{\partial v_x}{\partial y} \Big|_{y=0} = \frac{2\nu v_e}{\delta} = \frac{6\nu \langle v_x \rangle}{B\Delta(3 - \Delta)}$$

$$\begin{aligned} \frac{d}{dx} \int_0^\delta v_x (v_e - v_x) dy &= \frac{d}{dx} (v_e^2 \delta) \int_0^1 [(2u - u^2)(1 - 2u + u^2)] du \text{ with } u = y/\delta \\ &= \frac{d}{dx} (v_e^2 \delta) \int_0^1 [2u - 4u^2 + 2u^3 - u^2 + 2u^3 - u^4] du \\ &= \frac{d}{dx} (v_e^2 \delta) [1 - (4/3) + (2/4) - (1/3) + (2/4) - (1/5)] \\ &= B \langle v_x \rangle^2 \left[\frac{27 + 9\Delta}{(3 - \Delta)^3} \right] \frac{d\Delta}{dx} [2/15] \end{aligned}$$

$$\begin{aligned}
\frac{dv_e}{dx} \int_0^\delta (v_e - v_x) dy &= v_e \delta \frac{dv_e}{dx} \int_0^1 [1 - 2u + u^2] du \\
&= v_e \delta \frac{dv_e}{dx} [1 - 1 + (1/3)] du \\
&= 9 \langle v_x \rangle^2 B \frac{\Delta}{(3 - \Delta)^3} \frac{d\Delta}{dx} [1/3]
\end{aligned}$$

With these substitutions, Eq. 4.4-13 gives

$$\begin{aligned}
\frac{6\nu x}{\langle v_x \rangle B^2} &= B \langle v_x \rangle^2 \left[\frac{27 + 9\Delta}{(3 - \Delta)^3} \right] \frac{d\Delta}{dx} [2/15] + 9 \langle v_x \rangle^2 B \frac{\Delta}{(3 - \Delta)^3} \frac{d\Delta}{dx} [1/3] \\
&= \frac{B \langle v_x \rangle^2}{(3 - \Delta)^3} \frac{d\Delta}{dx} \left[\frac{54 + 18\Delta + 45\Delta}{15} \right] \\
&= \frac{B \langle v_x \rangle^2}{(3 - \Delta)^3} \frac{d\Delta}{dx} \left[\frac{54 + 63\Delta}{15} \right]
\end{aligned}$$

Multiplication of both members by $(5/3)\Delta(3 - \Delta)/(B \langle v_x \rangle^2)$ gives

$$10 \left(\frac{\nu}{\langle v_x \rangle B^2} \right) = \frac{6\Delta + 7\Delta^2}{(3 - \Delta)^2} \frac{d\Delta}{dx}$$

in agreement with Eq. 4C.1-4.

c. Integration of the last equation with the initial condition that $\Delta = 0$ at $x = 0$ gives

$$\frac{\nu x}{\langle v_x \rangle B^2} = \frac{1}{10} \int_0^\Delta \frac{6s + 7s^2}{(3 - s)^2} ds$$

From integral tables we get the formulas

$$\begin{aligned}
\int \frac{s}{(a + bs)^2} ds &= \frac{1}{b^2} \left[\ln(a + bs) + \frac{a}{a + bs} \right] + C \\
\int \frac{s^2}{(a + bs)^2} ds &= \frac{1}{b^3} \left[a + bx - 2a \ln(a + bx) - \frac{a^2}{a + bx} \right] + C
\end{aligned}$$

which yield the definite integrals

$$\begin{aligned}
\int_0^\Delta \frac{s}{(3 - s)^2} ds &= \left[\ln \left(\frac{3 - \Delta}{3} \right) + \frac{\Delta}{3 - \Delta} \right] \\
\int_0^\Delta \frac{s^2}{(3 - s)^2} ds &= \left[(3 - \Delta) - 6 \ln \left(\frac{3 - \Delta}{3} \right) - \frac{9}{3 - \Delta} \right]
\end{aligned}$$

and the solution

$$\begin{aligned}
 10 \frac{vx}{\langle v_x \rangle B^2} &= 6 \left[\ln \left(\frac{3-\Delta}{3} \right) + \frac{\Delta}{3-\Delta} \right] + 7 \left[(3-\Delta) - 6 \ln \left(\frac{3-\Delta}{3} \right) - \frac{9}{3-\Delta} \right] \\
 &= 7\Delta + 48 \ln \left(\frac{3-\Delta}{3} \right) + \frac{6\Delta}{3-\Delta} + 21 - \frac{63}{3-\Delta} \\
 &= 7\Delta + 48 \ln \left(\frac{3-\Delta}{3} \right) + \frac{6\Delta - 21(3-\Delta) - 63}{3-\Delta} \\
 &= 7\Delta + 48 \ln \left(\frac{3-\Delta}{3} \right) + \frac{27\Delta}{3-\Delta}
 \end{aligned}$$

in agreement with Eq. 4C.1-5.

d. Setting $\Delta = 1$ and $x = L_e$ in the last equation gives

$$\begin{aligned}
 L_e &= \frac{1}{10} \left[7 + 48 \ln(2/3) + \frac{27}{2} \right] \\
 &= (0.1) [7 - 19.462325 + 13.5] = 0.104
 \end{aligned}$$

in agreement with answer (d).

e. Application of Eq. 4.3-5 to the region $\delta(x) < y < B$, with v_y neglected so that $v = v_e$ there, gives

$$\frac{1}{2} \rho v_e^2 + \mathcal{P} = \text{constant}$$

Insertion of the result of part a gives

$$\frac{1}{2} \rho \langle v_x \rangle^2 \left(\frac{3}{3-\Delta} \right)^2 + \mathcal{P} = \text{constant}$$

Evaluating the constant at $x = 0$, where $\Delta = 0$ and $\mathcal{P} = \mathcal{P}_i$, we get

$$\frac{1}{2} \rho \langle v_x \rangle^2 \left(\frac{3}{3-\Delta} \right)^2 + \mathcal{P} = \frac{1}{2} \rho \langle v_x \rangle^2 + \mathcal{P}_0$$

or

$$\mathcal{P} - \mathcal{P}_0 = \frac{1}{2} \rho \langle v_x \rangle^2 \left[1 - \left(\frac{3}{3-\Delta} \right)^2 \right]$$

4C.2 Torsional oscillatory viscometer

a. The equation for the rotating bob in a vacuum is just "moment-of-inertia times angular acceleration equals the the sum of the torques." In mathematical terms this is

$$I \frac{d^2 \theta_R}{dt^2} = -k \theta_R$$

in which $-k \theta_R$ is the "restoring torque." The solution to this equation is

$$\theta_R = C_1 \cos \sqrt{\frac{k}{I}} t + C_2 \sin \sqrt{\frac{k}{I}} t = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

This states that the bob is oscillating with a frequency $\omega_0 = \sqrt{k/I}$, which is called the natural frequency.

b. If there is a fluid in the gap, the equation of motion of the bob must contain both the "restoring torque" and the torque exerted by the fluid in the θ direction on the solid surface that is perpendicular to the r direction:

$$I \frac{d^2 \theta_R}{dt^2} = -k \theta_R - (2\pi RL)(R)(-\tau_{r\theta}|_{r=R})$$

Here $(2\pi RL)(-\tau_{r\theta}|_{r=R})$ is the force, and R is the lever arm. Next one uses Eq. B.1-11 to replace the shear stress by the appropriate product of viscosity and velocity gradient. This gives Eq. 4C.2-1.

c. Equation 4C.2-3 is obtained from simplifying Eq. B.6-5.

d. The choice of the variable x is convenient, since $x = 0$ at the surface of the bob, and $x = 1$ is the inner surface of the cup. There are many ways to select the other dimensionless variables, but the choices we have made allow the viscosity to appear only in the parameter called M . Equations 4C.2-11 and 12 follow immediately.

e. From Eq. 4C.2-14 we get for the bob angular displacement

$$\begin{aligned} \theta_R &= \Re \left\{ \theta_R^o \exp(i\bar{\omega}\tau) \right\} = (\theta_{Rr}^o + i\theta_{Ri}^o) (\cos \bar{\omega}\tau + i \sin \bar{\omega}\tau) \\ &= \theta_{Rr}^o \cos \bar{\omega}\tau - \theta_{Ri}^o \sin \bar{\omega}\tau \end{aligned}$$

Here θ_{Rr}° and θ_{Ri}° are the real and imaginary parts of the complex amplitude θ_R° . The angular displacement of the bob can also be written in terms of an amplitude and a phase shift:

$$\theta_R = A \cos(\bar{\omega}\tau - \alpha) = A \cos \bar{\omega}\tau \cos \alpha - A \sin \bar{\omega}\tau \sin \alpha$$

When the two expressions for the bob displacement are equated, we

$$A = \sqrt{\theta_{Rr}^{\circ 2} + \theta_{Ri}^{\circ 2}} = |\theta_R^{\circ}| \quad \text{and} \quad \tan \alpha = \frac{\theta_{Ri}^{\circ}}{\theta_{Rr}^{\circ}} = \frac{\Im\{\theta_R^{\circ}\}}{\Re\{\theta_R^{\circ}\}}$$

i. When the hyperbolic functions are expanded in Taylor series and the terms arranged in powers of $1/M$, we get

$$\begin{aligned} \frac{\theta_{aR}^{\circ}}{\theta_R^{\circ}} &= \cosh \sqrt{\frac{i\bar{\omega}}{M}} + \frac{(1-\bar{\omega}^2)}{MAi\bar{\omega}} \frac{\sinh \sqrt{i\bar{\omega}/M}}{\sqrt{i\bar{\omega}/M}} \\ &= \left[1 + \frac{1}{2!} \left(\frac{i\bar{\omega}}{M} \right) + \frac{1}{4!} \left(\frac{i\bar{\omega}}{M} \right)^2 + \dots \right] - i \frac{(1-\bar{\omega}^2)}{MA\bar{\omega}} \left[1 + \frac{1}{3!} \left(\frac{i\bar{\omega}}{M} \right) \dots \right] \\ &= 1 - \frac{i}{M} \left[\frac{(1-\bar{\omega}^2)}{A\bar{\omega}} - \frac{\bar{\omega}}{2!} \right] + \frac{1}{M^2} \left[\frac{(1-\bar{\omega}^2)\bar{\omega}}{3!A\bar{\omega}} - \frac{\bar{\omega}^2}{4!} \right] + \dots \end{aligned}$$

From this, Eq. 4C.2-18 follows. The results of part (e) can then be used to get the amplitude and phase angle.

4C.3 Darcy's equation for flow through porous media.

(a) In Case 1, the equation of state is $\rho = \rho_0$. Hence, Eq. (4C.3-1) gives

$$(\nabla \cdot \mathbf{v}_0) = 0$$

Insertion of Eq. (4C.3-2) for \mathbf{v}_0 , with $\rho = \rho_0$ and $\mathbf{g} = -\nabla \hat{\Phi}$, gives

$$(\nabla \cdot [\nabla p + \rho_0 \nabla \hat{\Phi}]) = 0$$

or

$$\nabla^2 \mathcal{P} = 0$$

in which $\mathcal{P} = p + \rho_0 \hat{\Phi}$ is the "modified pressure" defined in Chapters 2 and 3 for systems with constant ρ .

(b) In Case 2, the equation of state is $\rho = \rho_0 e^{\beta p}$. Hence,

$$\nabla \rho = \rho_0 \beta e^{\beta p} \nabla p = \rho \beta \nabla p, \text{ so that } \rho \nabla p = \frac{1}{\beta} \nabla \rho$$

Expressing the divergence term of Eq. (4C.3-1) via Darcy's equation, we get

$$\begin{aligned} -(\nabla \cdot \rho \mathbf{v}_0) &= +\frac{\kappa}{\mu} (\nabla \cdot \rho [\nabla p - \rho \mathbf{g}]) \\ &= \frac{\kappa}{\mu} (\nabla \cdot [\rho \nabla p - \rho^2 \mathbf{g}]) \\ &= \frac{\kappa}{\mu} \left[\frac{1}{\beta} \nabla^2 \rho - (\nabla \cdot \rho^2 \mathbf{g}) \right] \end{aligned}$$

Inserting this result, multiplied by $\mu\beta/\kappa$, into the smoothed continuity equation, we get

$$\frac{\varepsilon \mu \beta}{\kappa} \frac{\partial \rho}{\partial t} = \nabla^2 \rho - \beta (\nabla \cdot \rho^2 \mathbf{g})$$

(c) In Case 3, the equation of state is $\rho = \rho_0 p$. Hence, $\nabla \rho = \rho_0 \nabla p$ and $\nabla p = (1/\rho_0) \nabla \rho$. Then Eqs. (4C.3-1,2), with the $\rho \mathbf{g}$ term neglected, give

$$\begin{aligned} \varepsilon \frac{\partial \rho}{\partial t} &= -(\nabla \cdot \rho [-\frac{\kappa}{\mu} \nabla p]) \\ &= +(\nabla \cdot \rho \frac{\kappa}{\mu \rho_0} \nabla \rho) \\ &= \frac{\kappa}{\mu \rho_0} (\nabla \cdot \rho \nabla \rho) \\ &= \frac{\kappa}{\mu \rho_0} (\nabla \cdot \nabla (\frac{1}{2} \rho^2)) \\ &= \frac{\kappa}{2\mu \rho_0} \nabla^2 \rho^2 \end{aligned}$$

Hence,

$$\frac{2\varepsilon\mu\rho_0}{\kappa} \frac{\partial\rho}{\partial t} = \nabla^2 \rho^2$$

(d) In Case 4, the equation of state is $\rho = \rho_0 p^m$, giving $p = (\rho/\rho_0)^{1/m}$. Thus,

$$\nabla p = \frac{1}{m} \rho_0^{-1/m} \rho^{(1/m)-1} \nabla \rho$$

whence

$$\rho v_0 = -\frac{\kappa}{\mu m} \rho_0^{-1/m} \rho^{1/m} \nabla \rho$$

so that

$$\begin{aligned} -(\nabla \cdot \rho v_0) &= +\frac{\kappa}{\mu m} \rho_0^{-1/m} (\nabla \cdot \rho^{1/m} \nabla \rho) \\ &= \frac{\kappa}{\mu m} \rho_0^{-1/m} (\nabla \cdot \left[\frac{m}{m+1} \nabla \rho^{(m+1)/m} \right]) \end{aligned}$$

With this result, Eq. 4C-1 gives

$$\left(\frac{(m+1)\varepsilon\mu\rho_0^{1/m}}{\kappa} \right) \frac{\partial\rho}{\partial t} = \nabla^2 \rho^{(m+1)/m}$$

4C.4 Radial flow through a porous medium

a. For the radial flow (in cylindrical coordinates) of an incompressible fluid Eqs. 4C.3-1 and 2 become

$$0 = -\frac{1}{r} \frac{d}{dr} (rv_{0r}) \quad \text{and} \quad v_{0r} = -\frac{\kappa d\mathcal{P}}{\mu dr}$$

Integrating the first of these equations gives $v_{0r} = C_1/r$. Substituting this into the second equation and integrating gives

$$C_1 \ln r + C_2 = -\frac{\kappa}{\mu} \mathcal{P} \quad \text{or} \quad D_1 \ln r + D_2 = -\mathcal{P}$$

where new constants of integration have been introduced. These constants are determined from the boundary conditions:

$$\text{B. C. 1:} \quad D_1 \ln R_1 + D_2 = -\mathcal{P}_1$$

$$\text{B. C. 2:} \quad D_1 \ln R_2 + D_2 = -\mathcal{P}_2$$

When the integration constants have been determined, the pressure distribution is found to be

$$\frac{\mathcal{P} - \mathcal{P}_1}{\mathcal{P}_2 - \mathcal{P}_1} = \frac{\ln(r/R_1)}{\ln(R_2/R_1)}$$

b. The velocity distribution is then given by using Darcy's law

$$v_{0r} = -\frac{\kappa d\mathcal{P}}{\mu dr} = -\frac{\kappa}{\mu} (\mathcal{P}_2 - \mathcal{P}_1) \frac{1}{\ln(R_2/R_1)} \frac{1}{r} = -\frac{\kappa (\mathcal{P}_2 - \mathcal{P}_1)}{\mu r \ln(R_2/R_1)}$$

c. The mass rate of flow through the system is

$$w = \rho |v_{0r}(R_1)| 2\pi R_1 h = \frac{2\pi \kappa h (\mathcal{P}_2 - \mathcal{P}_1) \rho}{\mu \ln(R_2/R_1)}$$

4D.1. Flow near an oscillating wall

The problem is to solve Eq. 4.1-1, with the initial condition that $v_x = 0$ at $t = 0$, and the boundary conditions that $v_x = 0$ at $y = \infty$ and $v_x = v_0 \cos \omega t$ at $y = 0$. When we take the Laplace transform of Eq. 4.1-1 and the boundary and initial conditions, we get

$$p\bar{v}_x = \nu \frac{d^2\bar{v}_x}{dy^2} \quad \text{with } \bar{v}_x = v_0 \frac{p}{p^2 + \omega^2} \text{ at } y = 0 \text{ and } \bar{v}_x = 0 \text{ at } y = \infty$$

This ordinary differential equation is easily solved with the boundary conditions to give

$$\bar{v}_x = v_0 \frac{p}{p^2 + \omega^2} \exp(\sqrt{p/\nu}y)$$

This may be inverted by using the convolution theorem, or else by consulting a table of transforms (see, for example, Formula #11, on p. 246 of Vol. 1 of A. Erdélyi, et al., *Table of Integral Transforms*, McGraw-Hill, New York (1954)). The use of the table leads directly to Eq. 4D.1-1.

4D.2 Start-up of laminar flow in a circular tube

a. The partial differential equation, initial condition, and boundary conditions are

$$\rho \frac{\partial v_z}{\partial t} = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} + \mu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right)$$

with $v_z(r, 0) = 0$, $v_z(0, t) = \text{finite}$, and $v_z(R, t) = 0$. We now introduce the following dimensionless variables

$$\phi = \frac{v_z}{(\mathcal{P}_0 - \mathcal{P}_L)R^2/4\mu L}; \quad \xi = \frac{r}{R} \quad \tau = \frac{\mu t}{\rho R^2}$$

Then Eq. 4D.2-1 is obtained, along with the initial and boundary conditions: $\phi(\xi, 0) = 0$, $\phi(0, \tau) = \text{finite}$, and $\phi(R, \tau) = 0$.

b. The asymptotic solution is obtained by setting the time derivative equal to zero and solving the ordinary differential equation with the boundary conditions. Then the partial differential equation for $\phi_t(\xi, \tau)$ is

$$\frac{\partial \phi_t}{\partial \tau} = \frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial \phi_t}{\partial \xi} \right)$$

with ancillary conditions: $\phi_t(\xi, 0) = \phi_\infty(\xi)$, $\phi_t(0, \tau) = \text{finite}$, and $\phi_t(1, \tau) = 0$. We now try a solution of the form $\phi_t(\xi, \tau) = \Xi(\xi)T(\tau)$. This leads to two ordinary differential equations

$$\frac{dT}{d\tau} = -\alpha^2 T \quad \text{and} \quad \frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{d\Xi}{d\xi} \right) + \alpha^2 \Xi = 0$$

which have as their solutions

$$T = C_0 \exp(-\alpha^2 \tau) \quad \text{and} \quad \Xi = C_1 J_0(\alpha \xi) + C_2 Y_0(\alpha \xi)$$

in which the C_s are constants, and J_0 and Y_0 are zero-order Bessel functions. Since Y_0 is not finite at the tube axis, we must set C_2 equal to zero. Since Ξ must be zero at the tube wall, this will occur only if

$J_0(\alpha) = 0$. This will happen only at $\alpha_1, \alpha_2, \alpha_3, \dots$, that is, at the zeroes of J_0 . Hence there are many solutions to the Ξ -equation that will satisfy the boundary conditions: $\Xi_n = C_{1n} J_0(\alpha_n \xi)$. Therefore, the general solution to the partial differential equation must be

$$\phi_t = \sum_{n=1}^{\infty} B_n \exp(-\alpha_n^2 \tau) J_0(\alpha_n \xi)$$

The constants B_n are to be determined from the initial condition,

$$(1 - \xi^2) = \sum_{n=1}^{\infty} B_n J_0(\alpha_n \xi)$$

This is done by multiplying both sides of the last equation by $J_0(\alpha_m \xi) \xi$ and integrating from 0 to 1:

$$\int_0^1 J_0(\alpha_m \xi) \xi (1 - \xi^2) d\xi = \sum_{n=1}^{\infty} B_n \int_0^1 J_0(\alpha_n \xi) J_0(\alpha_m \xi) \xi d\xi$$

Because of the orthogonality properties of the Bessel functions, the only term on the right side that contributes is the term for $m = n$. The integrals may then be evaluated using some standard relations for Bessel functions. This gives

$$\frac{4J_1(\alpha_m)}{\alpha_m^3} = B_m \cdot \frac{1}{2} [J_1(\alpha_m)]^2 \quad \text{from which} \quad B_m = \frac{8}{\alpha_m^3 J_1(\alpha_m)}$$

and this leads directly to Eq. 4D.2-2.

4D.3 Flows in the disk-and-tube system

a. If the tangential component of the velocity depends only on r and z , then the equation of motion simplifies to

$$\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (rv_\theta) \right) + \frac{\partial^2 v_\theta}{\partial z^2} = 0$$

We try a solution of the product form: $v_\theta(r, \theta) = f(r)g(z)$. When this is substituted into the partial differential equation above, and the resulting equation divided by $f(r)g(z)$ we get

$$\frac{1}{f} \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (rf) \right) = - \frac{1}{g} \frac{d^2 g}{dz^2}$$

The left side is a function of r only and the right side a function of z , only and therefore both sides must equal a constant; we call the constant $-c^2$. Then we have two ordinary differential equations to solve:

$$\frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (rf) \right) + c^2 f = 0 \quad \text{and} \quad \frac{d^2 g}{dz^2} - c^2 g = 0$$

The second equation has the solution $g = A \cosh cz + B \sinh cz$, where A and B are constants. The requirement that the velocity be zero at $z = L$, gives

$$g = -B \frac{\sinh[cL(1 - [z/L])]}{\cosh cL}$$

where B is a constant. The first equation can also be written as

$$\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} + \left(c^2 - \frac{1}{r^2} \right) f = 0$$

which is now recognized as a Bessel equation of first order, with the solution $f = MJ_1(cr) + NY_1(cr)$, where M and N are constants. But since Y_1 becomes infinite at $r = 0$, only the J_1 term is needed, Since $f =$

0 at $r = R$, we must have $J_1(c_n) = 0$. There are infinitely many c_n that satisfy this equation. Hence the solution is of the form

$$v_\theta(\xi, \zeta) = \sum_{n=1}^{\infty} B_n J_1(c_n \xi) \frac{\sinh c_n L (1 - \zeta)}{\cosh c_n L}$$

The coefficients B_n are determined from the initial condition

$$R\Omega\xi = \sum_{n=1}^{\infty} B_n J_1(c_n \xi)$$

This is done by multiplying the equation by $J_1(c_m \xi)\xi$ and integrating over ξ from 0 to 1 and making use of the orthogonality relation.

This gives

$$R\Omega \int_0^1 J_1(c_m \xi) \xi^2 d\xi = \sum_{n=1}^{\infty} B_n \int_0^1 J_1(c_n \xi) J_1(c_m \xi) \xi d\xi \quad \text{or}$$

$$R\Omega \frac{J_2(c_m)}{c_m} = B_m \cdot \frac{1}{2} [J_2(c_m)]^2 \quad \text{or} \quad B_m = \frac{2R\Omega}{c_m J_2(c_m)}$$

Thus the steady-state velocity distribution is

$$v_\theta(\xi, \zeta) = 2R\Omega \sum_{n=1}^{\infty} \frac{J_1(c_n \xi)}{c_n J_2(c_n)} \frac{\sinh c_n L (1 - \zeta)}{\cosh c_n L}$$

4D.4 Unsteady annular flows

For the tangential annular flow (part (b)), the equation of motion for $v_\theta(r, t)$ is

$$\rho \frac{\partial v_\theta}{\partial t} = \mu \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right]$$

with $v_\theta(r, 0) = 0$, $v_\theta(\kappa R, t) = \kappa R \Omega_i$, $v_\theta(R, t) = R \Omega_o$. We introduce these dimensionless variables:

$$\xi = \frac{r}{R}, \quad \tau = \frac{\mu t}{\rho R^2}; \quad \phi = \frac{v_\theta}{R(\Omega_o - \Omega_i)} \quad -\alpha = \frac{\Omega_i}{\Omega_o - \Omega_i}$$

Then the partial differential equation for $\phi(\xi, \tau)$ becomes

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial}{\partial \xi} \left[\frac{1}{\xi} \frac{\partial}{\partial \xi} (\xi \phi) \right]$$

with $\phi(\xi, 0) = 0$, $\phi(\kappa, \tau) = -\kappa\alpha$, $\phi(1, \tau) = 1 - \alpha$.

For the steady state (i.e., at infinite time), we have the solution

$$\phi_\infty = \left(\frac{1 - \alpha(1 - \kappa^2)}{1 - \kappa^2} \right) \xi - \left(\frac{\kappa^2}{1 - \kappa^2} \right) \frac{1}{\xi} \equiv A\xi - B\frac{1}{\xi}$$

Hence the time-dependent solution is

$$\phi(\xi, \tau) = \phi_\infty(\xi) - \phi_t(\xi, \tau)$$

Here $\phi_t(\xi, \tau)$ is the transient contribution, which satisfies the partial differential equation for $\phi(\xi, \tau)$, but with boundary and initial conditions: $\phi_t(\kappa, \tau) = \phi_t(1, \tau) = 0$ and $\phi_t(\xi, 0) = \phi_\infty(\xi)$

Application of the method of separation of variables with $\phi_t(\xi, \tau) = f(\xi)g(\tau)$ gives

$$\frac{1}{g} \frac{dg}{d\tau} = -b^2 = \frac{1}{f} \frac{d}{d\xi} \left(\frac{1}{\xi} \frac{d}{d\xi} (\xi f) \right)$$

where $-b^2$ is the separation constant. Thus we arrive at two ordinary differential equations which can be solved (the second-order equation is a Bessel function). Since this is a Sturm-Liouville equation, the complete solution is a sum of eigenfunctions multiplied by the appropriate exponential function:

$$\phi_t(\xi, \tau) = \sum_{n=1}^{\infty} C_n Z_1(b_n \xi) e^{-b_n^2 \tau}$$

in which $Z_1(b_n \xi)$ is the superposition of the two solutions to the Bessel equation:

$$Z_1(b_n \xi) = J_1(b_n \xi) Y_1(b_n \kappa) - J_1(b_n \kappa) Y_1(b_n \xi)$$

These functions satisfy the boundary conditions in that $Z_1(b_n \xi) = 0$ at $\xi = \kappa$. The conditions that $Z_1(b_n \xi) = 0$ at $\xi = 1$ determine the eigenvalues b_n . The C_n can then be determined from the initial

- i. The expressions for A and B
- ii. The defining equation for the b_n
- iii. The relation $J_0(x)Y_1(x) - J_1(x)Y_0(x) = -2/\pi x$

Then it may be shown that

$$Z_0(b_n) = -\frac{2}{\pi b_n} \frac{J_1(b_n \kappa)}{J_1(b_n)} \quad \text{and} \quad Z_0(b_n \kappa) = -\frac{2}{\pi b_n} \frac{1}{\kappa}$$

and also

$$C_n(\kappa, \alpha) = \frac{\pi J_1(b_n) [(1 - \alpha)J_1(b_n \kappa) + \alpha \kappa J_1(b_n)]}{[J_1^2(b_n \kappa) - J_1^2(b_n)]}$$

in which α is the dimensionless angular velocity. Then the complete expression for the transient behavior in tangential annular flow of a Newtonian fluid is

$$\phi(\xi, \tau) = \left(\frac{1 - \alpha(1 - \kappa^2)}{1 - \kappa^2} \right) \xi - \left(\frac{\kappa^2}{1 - \kappa^2} \right) \frac{1}{\xi} - \pi \sum_{n=1}^{\infty} \frac{J_1(b_n) [(1 - \alpha)J_1(b_n \kappa) + \alpha \kappa J_1(b_n)] Z_1(b_n \xi)}{[J_1^2(b_n \kappa) - J_1^2(b_n)]} e^{-b_n^2 \tau}$$

Complete tables are given in the original reference. as well as some typical velocity profiles. Also, a Laplace-transform solution is given for small times, for which the expression above converges too slowly to be of much value.

4D.5 Stream functions for three-dimensional flow.

(a) The divergence of the curl of \mathbf{A} is $\sum_i (\partial/\partial x_i) [\nabla \times \mathbf{A}]_i$ which gives, according to Eq. A.4-10,

$$\begin{aligned} \sum_i (\partial/\partial x_i) \sum_j \sum_k \varepsilon_{ijk} (\partial/\partial x_j) A_k &= \sum_i \sum_j \sum_k \varepsilon_{ijk} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\ &= \sum_k \sum_i \sum_{j \neq i} \varepsilon_{ijk} \left[\frac{\partial^2 A_k}{\partial x_i \partial x_j} - \frac{\partial^2 A_k}{\partial x_j \partial x_i} \right] = 0 \end{aligned}$$

Thus, the mass flux function $\rho \mathbf{v} = [\nabla \times \mathbf{A}]$ satisfies the continuity equation for steady flow (or for unsteady flow with constant density). Here use has been made of the relations $\varepsilon_{ijk} = -\varepsilon_{jik}$ and $\varepsilon_{iik} = \varepsilon_{ikk} = 0$ contained in Eq. A.2-15.

The divergence of the product expression for $\rho \mathbf{v}$ is

$$\begin{aligned} (\nabla \cdot [(\nabla \psi_1) \times (\nabla \psi_2)]) &= \sum_i \left(\delta_i \frac{\partial}{\partial x_i} \cdot \left[\sum_j \delta_j \frac{\partial \psi_1}{\partial x_j} \times \sum_k \delta_k \frac{\partial \psi_2}{\partial x_k} \right] \right) \\ &= \sum_i \sum_j \sum_k (\delta_i \cdot [\delta_j \times \delta_k]) \frac{\partial}{\partial x_i} \left(\frac{\partial \psi_1}{\partial x_j} \frac{\partial \psi_2}{\partial x_k} \right) \\ &= \sum_i \sum_j \sum_k (\delta_i \cdot \varepsilon_{jki} \delta_i) \frac{\partial}{\partial x_i} \left(\frac{\partial \psi_1}{\partial x_j} \frac{\partial \psi_2}{\partial x_k} \right) \\ &= \sum_i \sum_j \sum_k \varepsilon_{ijk} \partial_i (\partial_j \psi_1 \partial_k \psi_2) \\ &= \sum_i \sum_j \sum_k \varepsilon_{ijk} (\partial_{ij} \psi_1 \partial_k \psi_2 + \partial_j \psi_1 \partial_{ik} \psi_2) \\ &= \varepsilon_{123} (\partial_{12} \psi_1 \partial_3 \psi_2 + \partial_2 \psi_1 \partial_{13} \psi_2) \\ &\quad + \varepsilon_{231} (\partial_{23} \psi_1 \partial_1 \psi_2 + \partial_3 \psi_1 \partial_{21} \psi_2) \\ &\quad + \varepsilon_{312} (\partial_{31} \psi_1 \partial_2 \psi_2 + \partial_1 \psi_1 \partial_{32} \psi_2) \\ &\quad + \varepsilon_{132} (\partial_{13} \psi_1 \partial_2 \psi_2 + \partial_3 \psi_1 \partial_{12} \psi_2) \\ &\quad + \varepsilon_{321} (\partial_{32} \psi_1 \partial_1 \psi_2 + \partial_2 \psi_1 \partial_{31} \psi_2) \\ &\quad + \varepsilon_{213} (\partial_{21} \psi_1 \partial_3 \psi_2 + \partial_1 \psi_1 \partial_{23} \psi_2) \\ &= \partial_{12} \psi_1 \partial_3 \psi_2 (\varepsilon_{123} + \varepsilon_{213}) + \partial_2 \psi_1 \partial_{13} \psi_2 (\varepsilon_{123} + \varepsilon_{321}) \\ &\quad + \partial_{23} \psi_1 \partial_1 \psi_2 (\varepsilon_{231} + \varepsilon_{321}) + \partial_3 \psi_1 \partial_{21} \psi_2 (\varepsilon_{231} + \varepsilon_{132}) \\ &\quad + \partial_{31} \psi_1 \partial_2 \psi_2 (\varepsilon_{312} + \varepsilon_{132}) + \partial_1 \psi_1 \partial_{32} \psi_2 (\varepsilon_{312} + \varepsilon_{213}) \\ &= 0 \end{aligned}$$

Thus, the product expression for $\rho \mathbf{v}$ is divergenceless. Here Eq. A.2-15 and the symmetry relations $\partial_{ij} = \partial_{ji}$ have been used. Stream functions of this form have been used by several authors; see footnote 1 of §4.2.

(b) In each of the coordinate systems shown in Table 4.2-1, the two nontrivial velocity components for incompressible flow are the corresponding components of $\text{curl}(-\delta_3 \mathbf{A}_3)/\rho = \text{curl}(-\delta_3 \psi/h_3)$, as may be verified by use of Eqs. G, H and I in Tables A.7-1,2,3. Thus, for Cartesian coordinates with $v_z = 0$ and no z -dependence, $h_3 = h_z = 1$, and Eq. A.7-18 (with \mathbf{v} read as $\delta_3 \psi/h_3$) gives the velocity components

$$v_x = \text{curl}_x(-\mathbf{A})/\rho = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v_y = \text{curl}_y(-\mathbf{A})/\rho = +\frac{\partial \psi}{\partial x}$$

For cylindrical coordinates with $v_z = 0$ and no z -dependence, $h_3 = h_z = 1$ and Eq. A.7-18 (with \mathbf{v} read as $\delta_3 \psi/h_3$) gives the velocity components

$$v_r = \text{curl}_r(-\mathbf{A})/\rho = -\frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = \text{curl}_\theta(-\mathbf{A})/\rho = +\frac{\partial \psi}{\partial r}$$

For cylindrical coordinates with $v_\theta = 0$ and no θ -dependence, $h_3 = h_\theta = r$ and Eq. A.7-18 (with \mathbf{v} read as $\delta_3 \psi/h_3$) gives the velocity components

$$v_z = \text{curl}_z(-\mathbf{A})/\rho = -\frac{1}{r} \frac{\partial \psi}{\partial r} \quad \text{and} \quad v_r = \text{curl}_r(-\mathbf{A})/\rho = +\frac{1}{r} \frac{\partial \psi}{\partial z}$$

For spherical coordinates with $v_\phi = 0$ and no ϕ -dependence, $h_3 = h_\phi = r \sin \theta$ and Eq. A.7-18 (with \mathbf{v} read as $\delta_3 \psi/h_3$) gives the velocity components

$$v_r = \text{curl}_r(-\mathbf{A})/\rho = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi} \quad \text{and} \quad v_\theta = \text{curl}_\theta(\mathbf{A})/\rho = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}$$

(c) Consider two surfaces, $\psi_1(x_1, x_2, x_3) = C_1$ and $\psi_2(x_1, x_2, x_3) = C_2$, which intersect along a line \mathcal{L} . At each point on \mathcal{L} , the vectors $\nabla \psi_1$ and $\nabla \psi_2$ are normal to both surfaces, and the velocity vector $\mathbf{v} = [(\nabla \psi_1) \times (\nabla \psi_2)]$ is consequently tangent to \mathcal{L} . Thus, the intersection of any such pair of surfaces is a streamline. In Fig. 4.3-1, we may choose $\psi_1 = \Psi(r, \theta)$ and $\psi_2 = z$; the resulting streamlines in a plane of constant ψ_2 are shown in the figure.

(d) Read \mathbf{v} in Eq. A.5-4 as the vector \mathbf{A} whose curl is the local mass flux $\rho \mathbf{v}$. Then the net mass flow through S is

$$\int_S (\mathbf{n} \cdot [\nabla \times \mathbf{A}]) dS = \oint_C (\mathbf{t} \cdot \rho \mathbf{v}) dC$$

for steady flow, or for unsteady incompressible flow. A no-slip condition $\mathbf{v} = 0$ on C requires $[\nabla \times \mathbf{A}] = 0$ there, but this derivative condition does not require the vanishing of \mathbf{A} nor of the net mass flow.

5A.1 Pressure drop needed for laminar-turbulent transition.

The minimum value of $Re = 4w/(\pi D\mu)$ needed to produce turbulent flow in a long, smooth tube is about 2100. Poiseuille's law, Eq. 2.3-21, holds until this critical Re value, giving

$$w = \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)R^4\rho}{8\mu L} \text{ for } Re < Re_{crit}$$

Hence, the pressure gradient needed to initiate the laminar-turbulent transition is

$$\begin{aligned} |dp/dz| &= \frac{8\mu w}{\pi R^4 \rho} \\ &= \left(\frac{8\mu}{\pi R^4}\right) \left(\frac{\pi D\mu Re_{crit}}{4}\right) \\ &\approx \frac{4\mu^2}{\rho R^3} (2.1 \times 10^3) \end{aligned}$$

The other specifications for this problem are:

$$\mu = 18.3 \text{ cp} = 0.183 \text{ g/cm}\cdot\text{s}; \quad \rho = 1.32 \text{ g/cm}^3; \quad R = 2.67 \text{ cm}$$

The pressure gradient required to initiate turbulence at the given conditions is then:

$$\begin{aligned} \left|\frac{dp}{dz}\right| &\approx \frac{(4)(0.183 \text{ g/cm}\cdot\text{s})^2}{(1.32 \text{ g/cm}^3)(2.67 \text{ cm})^3} (2.1 \times 10^3) \\ &\approx (11.2 \text{ dyne/cm}^3)(0.1 \text{ Pa}/(\text{dyne/cm}^2))(10^5 \text{ cm/km}) \\ &\approx 1.1 \times 10^5 \text{ Pa/km} \\ &\approx (1.1 \times 10^5 \text{ Pa/km})(1.45 \times 10^{-4} \text{ psi/Pa})(1.602 \text{ km/mile}) = 26 \text{ psi/mile} \end{aligned}$$

5A.2 Velocity distribution in turbulent pipe flow.

a. Application of Eq. 5.4-4 gives

$$\begin{aligned}\tau_0 &= \frac{(p_0 - p_L)R}{2L} = \left(\frac{(1.0 \text{ psi})(0.25 \text{ ft})}{(2)(5280 \text{ ft})} \right) = 2.367 \times 10^{-5} \text{ psi} \\ &= (2.367 \times 10^{-5} \text{ psi})(6.8947 \times 10^3 \text{ Pa/psi}) = 0.1633 \text{ Pa}\end{aligned}$$

b. For use of Fig. 5.5-3 we need the following additional values:

$$\mu = 1.0019 \times 10^{-3} \text{ Pa}\cdot\text{s}$$

$$\nu = 1.0037 \times 10^{-2} \text{ cm}^2/\text{s} = 1.0037 \times 10^{-6} \text{ m}^2/\text{s}$$

$$\rho = 0.9992 \text{ g/cm}^3 = 0.9992 \times 10^3 \text{ kg/m}^3$$

$$v_* = \sqrt{\tau_0/\rho} = \sqrt{(0.1633 \text{ kg/m}\cdot\text{s}^2)/(0.9992 \times 10^3 \text{ kg/m}^3)} = 0.01278 \text{ m/s}$$

$$v_*/\nu = (0.01278 \text{ m/s})/(1.0037 \times 10^{-6} \text{ m}^2/\text{s}) = 1.273 \times 10^4 \text{ m}^{-1}$$

$$R = (3 \text{ in})/(39.37 \text{ in/m}) = 0.0762 \text{ m}$$

At the tube center, $y^+ = Rv_*/\nu = (0.0762)(1.27 \times 10^4) = 970$ and Fig. 5.5-3 gives $v_*|_{y=R} = 22.7$. Consequently,

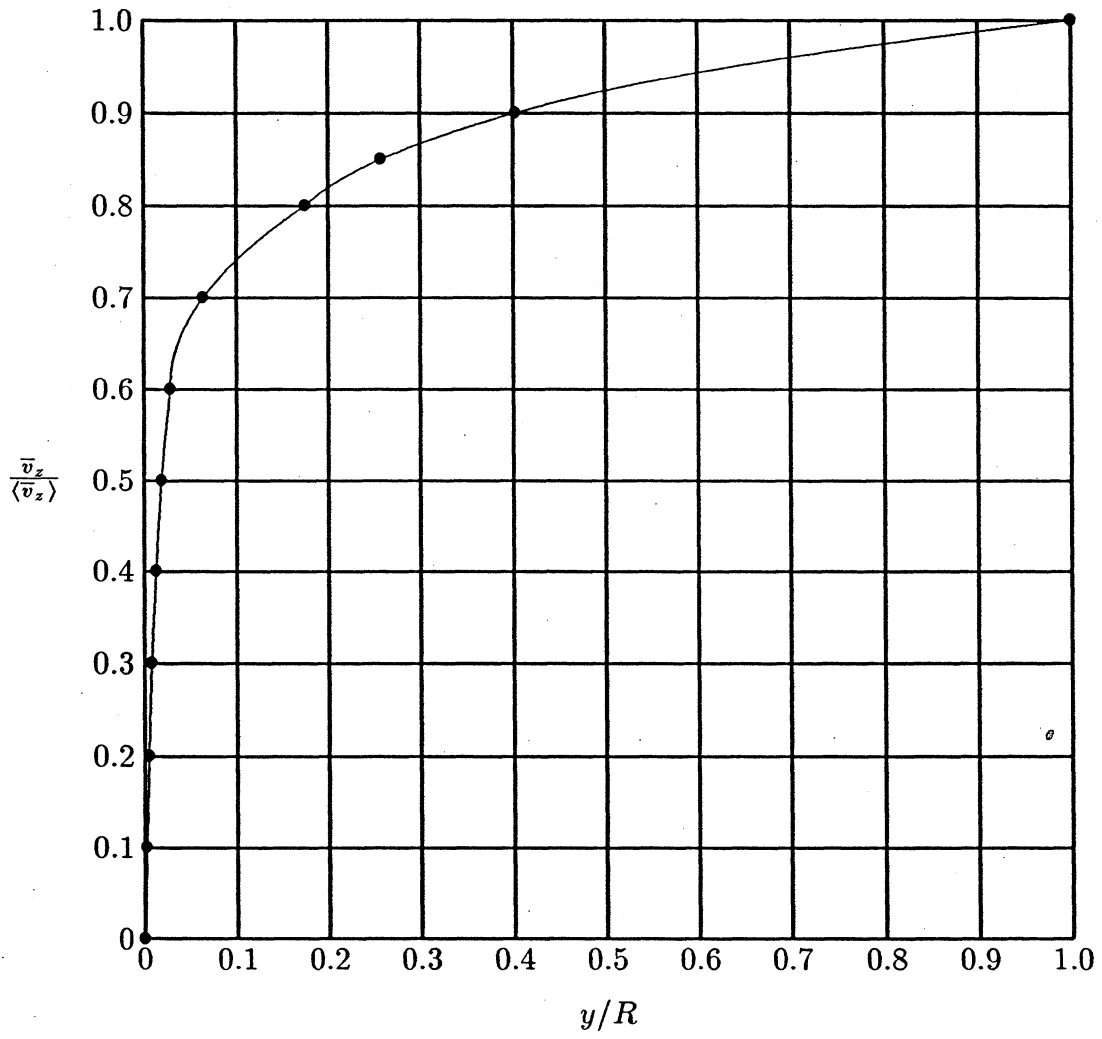
$$\bar{v}_{z\max} = 22.7v_* = 0.290 \text{ m/s}, \quad v^+ = 22.7\bar{v}_z/\bar{v}_{z\max}, \quad \text{and} \quad y/R = s^+/970$$

We can now tabulate the time-smoothed velocity profile. Asterisked values of $\bar{v}_z/\bar{v}_{z\max}$ are added here to give a better calculation of the mass flow rate in part (d).

$\bar{v}_z/\bar{v}_{z\max}$	v^+	y^+ (from Fig. 5.5-3)	y/R	r/R
0.0	0.0	0.0	0.0	1.0
0.1	2.27	2.27	0.0023	0.9977
0.2	4.54	4.55	0.0047	0.9953
0.3*	6.81	7.3	0.0075	0.9925
0.4	9.08	12.0	0.0124	0.9876
0.5*	11.35	18.0	0.0186	0.9814
0.6*	13.62	27.5	0.0284	0.9716
0.7	15.89	62.0	0.064	0.936
0.8*	18.16	170	0.175	0.825
0.85	19.30	250	0.258	0.742
0.9*	20.43	392	0.404	0.596
1.00	22.7	970	1.00	0.000

c. See the graph on the following page.

Velocity profile in Problem 5A.2



d. The estimate $\langle \bar{v}_z \rangle \approx 0.85\bar{v}_{z\max}$ gives $\langle \bar{v}_z \rangle \approx (0.85)(0.290 \text{ m/s})$. Hence,

$$\text{Re} = \frac{D\langle \bar{v}_z \rangle}{\nu} = \frac{(2 \times 0.0762 \text{ m})(0.85)(0.290 \text{ m/s})}{(1.0037 \times 10^{-6} \text{ m}^2/\text{s})} = 3.7 \times 10^6$$

so the flow is certainly turbulent.

e. The mean value of v^+ over the flow cross-section is

$$\langle v^+ \rangle = \int_0^1 v^+ d(r/R)^2$$

Applying trapezoidal quadrature to the values calculated in (b), we get

$$\begin{aligned} \int_0^1 v^+ d(r/R)^2 &= \left(\frac{0 + 2.27}{2} \right) [1.0000^2 - 0.9977^2] \\ &+ \left(\frac{2.27 + 4.54}{2} \right) [0.9977^2 - 0.9953^2] \\ &+ \left(\frac{4.54 + 6.81}{2} \right) [0.9953^2 - 0.9925^2] \\ &+ \left(\frac{6.81 + 9.08}{2} \right) [0.9925^2 - 0.9876^2] \\ &+ \left(\frac{9.08 + 11.35}{2} \right) [0.9876^2 - 0.9814^2] \\ &+ \left(\frac{11.35 + 13.62}{2} \right) [0.9814^2 - 0.9716^2] \\ &+ \left(\frac{13.62 + 15.89}{2} \right) [0.9716^2 - 0.936^2] \\ &+ \left(\frac{15.89 + 18.16}{2} \right) [0.936^2 - 0.825^2] \\ &+ \left(\frac{18.16 + 19.30}{2} \right) [0.825^2 - 0.742^2] \\ &+ \left(\frac{19.30 + 20.43}{2} \right) [0.742^2 - 0.596^2] \\ &+ \left(\frac{20.43 + 22.7}{2} \right) [0.596^2 - 0.000^2] \\ &= 18.8 \text{ m/s} \end{aligned}$$

Hence,

$$\langle \bar{v}_z \rangle = \langle v^+ \rangle v_* = (18.8)(0.01278 \text{ m/s}) = 0.241 \text{ m/s}$$

and

$$\begin{aligned} w &= \pi R^2 \rho \langle \bar{v}_z \rangle \\ &= \pi (0.0762 \text{ m})^2 (999.2 \text{ kg/m}^3) (0.241 \text{ m/s}) \\ &= 4.4 \text{ kg/s} = 9.7 \text{ lb}_m/\text{s} \end{aligned}$$

5B.1 Average flow velocity in turbulent tube flow

a. The ratio of average to maximum velocity is obtained as follows for the power-law expression:

$$\begin{aligned} \frac{\langle \bar{v}_z \rangle}{\bar{v}_{z,\max}} &= \frac{\int_0^{2\pi} \int_0^R (\bar{v}_z / \bar{v}_{z,\max}) r dr d\theta}{\int_0^{2\pi} \int_0^R r dr d\theta} = \frac{2}{R^2} \int_0^R \left(1 - \frac{r}{R}\right)^{1/n} r dr \\ &= 2 \int_0^1 (1 - \xi)^{1/n} \xi d\xi = 2 \int_0^1 \zeta^{1/n} (1 - \zeta) d\zeta \\ &= 2 \left(\frac{\zeta^{(1/n)+1}}{(1/n)+1} - \frac{\zeta^{(1/n)+2}}{(1/n)+2} \right) \Bigg|_0^1 \\ &= 2 \frac{n[(2n+1) - (n+1)]}{(n+1)(2n+1)} = \frac{2n^2}{(n+1)(2n+1)} \end{aligned}$$

b. For the logarithmic profile we have

$$\begin{aligned} \frac{\langle \bar{v}_z \rangle}{\bar{v}_{z,\max}} &= \frac{v_* \int_0^{2\pi} \int_0^R (\bar{v}_z / v_*) r dr d\theta}{\bar{v}_{z,\max} \int_0^{2\pi} \int_0^R r dr d\theta} = \frac{v_*}{\bar{v}_{z,\max}} \frac{2}{R^2} \int_0^R \left(\frac{1}{\kappa} \ln \frac{yv_*}{\nu} + \lambda \right) r dr \\ &= \frac{v_*}{\bar{v}_{z,\max}} \frac{2}{R^2} \int_0^R \left(\frac{1}{\kappa} \ln \frac{yv_*}{\nu} + \lambda \right) (R - y) dy \quad \text{where } y = R - r \\ &= \frac{2v_*}{\bar{v}_{z,\max}} \left(\frac{\nu}{Rv_*} \right)^2 \int_0^{Rv_*/\nu} \left(\frac{1}{\kappa} \ln y^+ + \lambda \right) \left(\frac{Rv_*}{\nu} - y^+ \right) dy^+ \quad (y^+ = \frac{yv_*}{\nu}) \\ &= \frac{2v_*}{\bar{v}_{z,\max}} \left(\frac{\nu}{Rv_*} \right)^2 \left[\frac{1}{\kappa} \frac{Rv_*}{\nu} (y^+ \ln y^+ - y^+) - \frac{1}{\kappa} \left(\frac{1}{2} y^{+2} \ln y^+ - \frac{1}{4} y^{+2} \right) \right. \\ &\quad \left. + \lambda \frac{Rv_*}{\nu} y^+ - \lambda \frac{1}{2} y^{+2} \right]_0^{Rv_*/\nu} \\ &= \frac{2v_*}{\bar{v}_{z,\max}} \left[\frac{1}{\kappa} \left(\ln \frac{Rv_*}{\nu} - 1 \right) - \frac{1}{\kappa} \left(\frac{1}{2} \ln \frac{Rv_*}{\nu} - \frac{1}{4} \right) + \frac{1}{2} \lambda \right] \\ &= \frac{v_*}{\bar{v}_{z,\max}} \left[\frac{1}{\kappa} \ln \frac{Rv_*}{\nu} - \frac{3}{2} \frac{1}{\kappa} + \lambda \right] \end{aligned}$$

In the above we have used the fact that in the limit as x goes to zero, $x \ln x$ vanishes.

5B.2 Mass flow rate in a turbulent circular jet

a. Immediately starting to differentiate the velocity components with respect to the position coordinates is *not* the way to solve this problem. It is much easier to solve if one introduces some abbreviations in order to minimize writing:

$$Z = \frac{\bar{v}_z}{v^{(t)}} \quad R = \frac{\bar{v}_r}{v^{(t)}} \quad C_3 = C \quad u = \frac{Cr}{z} \quad [] = \left[1 + \frac{1}{4}u^2\right]$$

In terms of these quantities, the velocity components are

$$Z = \frac{2C^2}{z[]^2} \quad \text{and} \quad R = \frac{C(u - \frac{1}{3}u^3)}{z[]^2}$$

Then we can calculate the derivatives appearing in the equation of continuity:

$$\begin{aligned} \frac{\partial Z}{\partial z} &= -\frac{2C^2}{z^2[]^2} - 2\frac{2C^2}{z[]^3} \left(\frac{1}{2}u\right) \left(-\frac{Cr}{z^2}\right) = -\frac{2C^2}{z^2[]^2} + \frac{2C^2u^2}{z^2[]^3} \\ &= -\frac{2C^2}{z^2[]^3} \left(\left(1 + \frac{1}{4}u^2\right) - u^2\right) = -\frac{2C^2}{z^2[]^3} \left(1 - \frac{3}{4}u^2\right) \end{aligned}$$

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} (rR) &= \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{u^2 - \frac{1}{4}u^4}{[]^2} \right) = \frac{1}{r} \left(\frac{(2u - u^3)}{[]^2} \left(\frac{C}{z}\right) - 2\left(u^2 - \frac{1}{4}u^4\right) \frac{\frac{1}{2}u}{[]^3} \left(\frac{C}{z}\right) \right) \\ &= \frac{1}{r[]^3} \left(\frac{C}{z}\right) \left(\left(1 + \frac{1}{4}u^4\right)(2u - u^3) - u\left(u^2 - \frac{1}{4}u^4\right) \right) = \frac{2C^2}{z^2[]^3} \left(1 - \frac{3}{4}u^2\right) \end{aligned}$$

Thus the equation of continuity is satisfied. Next we get the expressions for the terms on the left side of the equation of motion:

$$R \frac{\partial Z}{\partial r} = \left(\frac{C(u - \frac{1}{4}u^3)}{z[]^2} \right) \left(-\frac{2C^3u}{z^2[]^3} \right) = -\frac{2C^4u^2(1 - \frac{1}{4}u^2)}{z^3[]^5}$$

$$Z \frac{\partial Z}{\partial z} = \left(\frac{2C^2}{z[]^2} \right) \left(-\frac{2C^2(1 - \frac{3}{4}u^2)}{z^2[]^3} \right) = -\frac{4C^4(1 - \frac{3}{4}u^2)}{z^3[]^5}$$

$$\begin{aligned} R \frac{\partial Z}{\partial r} + Z \frac{\partial Z}{\partial z} &= -\frac{4C^4(1 - \frac{1}{4}u^2 - \frac{1}{8}u^4)}{z^3[]^5} = -\frac{4C^4(1 + \frac{1}{4}u^2)(1 - \frac{1}{2}u^2)}{z^3[]^5} \\ &= -\frac{4C^3u(1 - \frac{1}{2}u^2)}{rz^2[]^4} \end{aligned}$$

The term on the right side is

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial Z}{\partial r} \right) &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \left(-\frac{2C^3u}{z^2[]^3} \right) \right) = -\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{2C^2u^2}{z[]^3} \right) \\ &= -\frac{1}{r} \left(\frac{4C^2u}{z[]^3} \cdot \frac{C}{z} - \frac{3C^2u^3}{z[]^4} \cdot \frac{C}{z} \right) = -\frac{1}{r} \left(\frac{4C^3u(1 - \frac{1}{2}u^2)}{z^2[]^4} \right) \end{aligned}$$

Therefore the equation of motion is satisfied.

b. The mass flow rate is

$$\begin{aligned} w(z) &= \int_0^{2\pi} \int_0^\infty \rho \bar{v}_z(r, z) r dr d\theta = 2\pi\rho \cdot \frac{2C_3^2 v^{(t)}}{z} \int_0^\infty \frac{r dr}{\left[1 + \frac{1}{4}(C_3 r/z)^2 \right]^2} \\ &= 2\pi\rho \cdot \frac{2C_3^2 v^{(t)}}{z} \cdot \frac{1}{2} \int_0^\infty \frac{ds}{\left[1 + \frac{1}{4}(C_3/z)^2 s \right]^2} \\ &= 2\pi\rho \cdot \frac{2C_3^2 v^{(t)}}{z} \cdot \frac{1}{2} \left(-\frac{1}{\frac{1}{4}(C_3/z)^2 \cdot \left[1 + \frac{1}{4}(C_3/z)^2 s \right]} \right) \Bigg|_0^\infty \\ &= 2\pi\rho \cdot \frac{2C_3^2 v^{(t)}}{z} \cdot \frac{1}{2} \cdot \frac{4z^2}{C_3^2} = 8\pi\rho v^{(t)} z \end{aligned}$$

5B.3 The eddy viscosity expression in the viscous sublayer

We start with Eq. 5.3-13 in dimensionless form

$$v^+ = y^+ \left[1 - \frac{1}{2} \left(\frac{v}{v_* B} \right) y^+ - \frac{1}{4} \left(\frac{y^+}{14.5} \right)^3 + \dots \right]$$

from which

$$\frac{dv^+}{dy^+} = 1 - \left(\frac{v}{v_* B} \right) y^+ - \left(\frac{y^+}{14.5} \right)^3 + \dots$$

For slit flow with thickness $2B$, we have

$$\bar{\tau}_{yx} = \bar{\tau}_{yx}^{(v)} + \bar{\tau}_{yx}^{(t)} = -\tau_0 \left(1 - \frac{y}{B} \right) = -(\mu + \mu^{(t)}) \frac{d\bar{v}_x}{dy}$$

where the definition of turbulent viscosity in Eq. 5.4-1 has been used. If now we write this last equation in dimensionless form, we have

$$\frac{\bar{\tau}_{yx}}{\rho v_*^2} = -1 + \left(\frac{v}{v_* B} \right) y^+ = - \left(1 + \frac{\mu^{(t)}}{\mu} \right) \frac{dv^+}{dy^+}$$

Substituting the dimensionless velocity gradient obtained from Eq. 5.3-13 into this expression gives

$$\begin{aligned} -1 + \left(\frac{v}{v_* B} \right) y^+ &= -1 + \left(\frac{v}{v_* B} \right) y^+ + \left(\frac{y^+}{14.5} \right)^3 + \dots \\ &\quad - \frac{\mu^{(t)}}{\mu} + \frac{\mu^{(t)}}{\mu} \left(\frac{v}{v_* B} \right) y^+ + \frac{\mu^{(t)}}{\mu} \left(\frac{y^+}{14.5} \right)^3 + \dots \end{aligned}$$

Terms 1 and 2 on the left exactly balance terms 1 and 2 on the right. Term 3 on the right will exactly cancel term 4 if the expression for the turbulent viscosity given in Eq. 5.4-2 is used. With the same substitution, terms 5 and 6 on the right side are of higher order and can be neglected in the vicinity of the wall.

5C.1 Two-dimensional turbulent jet

a. The total flow of z momentum in the jet is given by

$$J = \int_0^W \int_{-\infty}^{+\infty} \rho \bar{v}_z^2 dx dy = W \rho \bar{v}_{z,\max}^2 z \int_{-\infty}^{+\infty} f^2 d\xi$$

Since J is constant for all values of z , we conclude that $\bar{v}_{z,\max} \propto 1/\sqrt{z}$.

b. By integrating one of the stream-function equations we get

$$\psi \propto \int \bar{v}_z dx \propto \bar{v}_{z,\max} \int f dx \propto z^{-1/2} \int f d\xi \cdot z \propto z^{1/2}$$

The momentum flow $J [=] Lm/t^2$ and $\sqrt{J/\rho} [=] L^2/t$. But $\psi [=] L^2/t$ also. This suggests that

$$\psi(x, z) = \sqrt{\frac{J}{\rho}} \sqrt{\frac{z}{W}} F(\xi)$$

is the only form that can be put together for the stream function.

c. According to Eq. 5.4-3, the kinematic viscosity is the product of b and $\bar{v}_{z,\max}$, in which b has dimensions of length and must be some function of z . This leads to the fact that the kinematic viscosity (which has dimensions of L^2/t) can be constructed thus:

$$\psi(x, z) \propto \sqrt{\frac{J}{\rho}} \sqrt{\frac{z}{W}} F(\xi) \nu^{(t)} \propto z \cdot \frac{1}{\sqrt{z}} \sqrt{\frac{J}{\rho}} \sqrt{\frac{1}{W}}$$

The third factor has been included in order to make the dimensions come out correctly. The dimensionless constant λ is then included, but this quantity must be determined experimentally.

d. The z component of the equation of motion is

$$\bar{v}_x \frac{\partial \bar{v}_z}{\partial x} + \bar{v}_z \frac{\partial \bar{v}_z}{\partial z} = \frac{\partial}{\partial x} \left(\nu^{(t)} \frac{\partial \bar{v}_z}{\partial x} \right)$$

or, in terms of the stream function,

$$-\frac{\partial \psi}{\partial z} \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial x \partial z} = -\frac{\partial}{\partial x} \left(v^{(t)} \frac{\partial^2 \psi}{\partial x^2} \right)$$

Inserting the expressions for dimensionless stream function from (b) and the kinematic viscosity from (c), we get

$$-\left(\frac{\partial}{\partial z} \sqrt{z} F \right) \left(\sqrt{z} \frac{\partial^2 F}{\partial x^2} \right) + \left(\sqrt{z} \frac{\partial F}{\partial x} \right) \left(\frac{\partial}{\partial x} \frac{\partial}{\partial z} \sqrt{z} F \right) = -\lambda \frac{\partial}{\partial x} \left(\sqrt{z} \frac{\partial^2 \psi}{\partial x^2} \sqrt{z} F \right)$$

where f is a function of ξ . We next convert the derivatives with respect to x and z to derivatives with respect to ξ (indicated by primes), to get

$$\begin{aligned} -\left(\frac{F}{2\sqrt{z}} - \sqrt{z} F' \frac{\xi}{z} \right) \left(\frac{\sqrt{z}}{z^2} F'' \right) \\ + \left(\frac{\sqrt{z}}{z} F' \right) \left(\frac{1}{2\sqrt{z} z} F' - \frac{\sqrt{z}}{z^2} \{ \xi F'' + F' \} \right) = -\lambda \frac{z}{z^3} F''' \end{aligned}$$

Multiplying by z^2 and canceling two terms in the equation gives

$$\frac{1}{2} F F'' + \frac{1}{2} F'^2 = \lambda F'''$$

When we change to a new variable $\eta = \xi/4\lambda$, this equation becomes

$$F \frac{d^2 F}{d\eta^2} + \left(\frac{dF}{d\eta} \right)^2 = \frac{1}{2} \frac{d^3 F}{d\eta^3} \quad \text{or} \quad \frac{d}{d\eta} \left(F \frac{dF}{d\eta} \right) = \frac{1}{2} \frac{d^3 F}{d\eta^3}$$

f. The final result in (d) can be integrated at once to give

$$F \frac{dF}{d\eta} = \frac{1}{2} \frac{d^2 F}{d\eta^2} + C' \quad \text{or} \quad \frac{d}{d\eta} F^2 = \frac{d^2 F}{d\eta^2} + 2C'$$

But at $\eta = 0$, both F and its second derivative are zero so that $C' = 0$.

g. Integration of the result in (f) gives

$$F^2 = \frac{dF}{d\eta} + C^2 \quad \text{or} \quad \frac{dF}{d\eta} = F^2 - C^2$$

h. Integrating this last result gives

$$\int_0^F \frac{dF}{F^2 - C^2} = \int_0^\eta d\eta \quad \text{or} \quad -\frac{1}{C} \operatorname{arctanh} \frac{F}{C} = \eta \quad \text{or} \quad F = -C \tanh C\eta$$

Then the time-smoothed axial velocity is

$$\begin{aligned} \bar{v}_z(x, z) &= -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x} \left(\sqrt{\frac{Jz}{\rho W}} F(\xi) \right) \\ &= -\sqrt{\frac{Jz}{\rho W}} \frac{dF}{d\xi} \frac{1}{z} = -\sqrt{\frac{J}{\rho W}} \frac{dF}{d\eta} \frac{1}{4\lambda} \\ &= \sqrt{\frac{J}{\rho Wz}} \frac{C^2}{4\lambda} \operatorname{sech}^2 C\eta = \sqrt{\frac{J}{\rho Wz}} \frac{C^2}{4\lambda} \operatorname{sech}^2 \frac{Cx}{4\lambda z} \end{aligned}$$

i. The mass rate of flow at plane z is

$$\begin{aligned} w &= \int_0^W \int_{-\infty}^{+\infty} \rho \bar{v}_z dx dy \\ &= W\rho \sqrt{\frac{J}{\rho Wz}} \frac{C^2}{4\lambda} \int_{-\infty}^{+\infty} \operatorname{sech}^2 C\eta d\eta \cdot 4\lambda z \\ &= W\rho z \sqrt{\frac{J}{\rho Wz}} C \cdot 2 \tanh x \Big|_0^\infty = 2CW\rho z \sqrt{\frac{J}{\rho Wz}} \\ &= 2\sqrt[3]{3\lambda} \sqrt{J\rho Wz} \end{aligned}$$

Thus we see that the mass rate of flow increases with the distance from the exit of the slit. This is a result of the fact that additional fluid is dragged along by the jet.

5C.2 Axial turbulent flow in an annulus

a. We divide the annular region into two parts, depending on whether or not r is less than or greater than bR :

Region <: $aR < r < bR$

$$\text{stress at inner wall} = |\tau_{rz}| = \frac{(\mathfrak{P}_0 - \mathfrak{P}_L)R}{2L} \left(\frac{b^2 - a^2}{a} \right)$$

Region >: $bR < r < R$

$$\text{stress at outer wall} = |\tau_{rz}| = \frac{(\mathfrak{P}_0 - \mathfrak{P}_L)R}{2L} (1 - b^2)$$

To get the magnitudes of the wall shear stresses, we use the expression given in Eq. 2.4-4, which is valid for laminar and turbulent flow alike. Then, the friction velocity in the two regions is

$$\text{Region <: } v_*^< = \sqrt{\frac{\tau_0^<}{\rho}} = \sqrt{\frac{(\mathfrak{P}_0 - \mathfrak{P}_L)R}{2L\rho} \left(\frac{b^2 - a^2}{a} \right)} = v_{**} \sqrt{\frac{b^2 - a^2}{a}}$$

$$\text{Region >: } v_*^> = \sqrt{\frac{\tau_0^>}{\rho}} = \sqrt{\frac{(\mathfrak{P}_0 - \mathfrak{P}_L)R}{2L\rho} (1 - b^2)} = v_{**} \sqrt{1 - b^2}$$

where $v_{**} = \sqrt{(\mathfrak{P}_0 - \mathfrak{P}_L)R/2L\rho}$. Then the velocity distribution in the two regions may be obtained from Eq. 5.3-3:

$$\text{Region <: } \frac{\bar{v}_z^<}{v_*^<} = \frac{1}{\kappa^<} \ln \left[\frac{(r - aR)v_*^<}{\nu} \right] + \lambda^< = \frac{1}{\kappa^<} \ln \left(\frac{yv_*^<}{\nu} \right) + \lambda^<$$

$$\text{Region >: } \frac{\bar{v}_z^>}{v_*^>} = \frac{1}{\kappa^>} \ln \left[\frac{(R - r)v_*^>}{\nu} \right] + \lambda^> = \frac{1}{\kappa^>} \ln \left(\frac{yv_*^>}{\nu} \right) + \lambda^>$$

y being defined as $y = r - aR$ in Region < and $y = R - r$ in Region >.

b. The continuity of the velocity profile at $r = bR$ gives a relation between $\lambda^<$ and $\lambda^>$:

$$\frac{1}{\kappa^<} \ln \left[\left(\frac{Rv_{**}}{v} \right) (b-a) \sqrt{\frac{b^2 - a^2}{a}} \right] + \lambda^< = \frac{1}{\kappa^>} \ln \left[\left(\frac{Rv_{**}}{v} \right) (1-b) \sqrt{1-b^2} \right] + \lambda^>$$

c. The mass rate of flow through the annulus is

$$w = 2\pi\rho \int_{aR}^{bR} \bar{v}_z^< r dr + 2\pi\rho \int_{bR}^R \bar{v}_z^> r dr$$

The first integral is:

$$\begin{aligned} & 2\pi\rho v_*^< \int_{aR}^{bR} \left[\frac{1}{\kappa^<} \ln \left(\frac{yv_*^<}{v} \right) + \lambda^< \right] r dr \\ &= 2\pi\rho v_*^< \int_0^{(b-a)R} \left[\frac{1}{\kappa^<} \ln \left(\frac{yv_*^<}{v} \right) + \lambda^< \right] (y + aR) dy \\ &= 2\pi\rho v_*^< \left(\frac{v}{v_*^<} \right)^2 \int_0^{(b-a)(Rv_*^</v)} \left(\frac{1}{\kappa^<} \ln x + \lambda^< \right) \left(x + \frac{aRv_*^<}{v} \right) dx \\ &= 2\pi\rho v_*^< \left(\frac{v}{v_*^<} \right)^2 \left[\frac{1}{\kappa^<} \left(\frac{x^2}{2} \ln x - \frac{x^2}{4} \right) + \frac{1}{\kappa^<} \left(\frac{aRv_*^<}{v} \right) (x \ln x - x) \right. \\ &\quad \left. + \frac{1}{2} \lambda^< x^2 + \lambda^< \left(\frac{aRv_*^<}{v} \right) x \right] \Bigg|_0^{(b-a)(Rv_*^</v)} \\ &= \pi R^2 \rho v_*^< \left[\frac{1}{\kappa^<} \left[(b-a)^2 + 2a(b-a) \right] \ln(b-a) \left(\frac{Rv_*^<}{v} \right) \right. \\ &\quad \left. + \lambda^< \left[(b-a)^2 + 2a(b-a) \right] - \frac{1}{\kappa^<} \left[\frac{1}{2} (b-a)^2 + 2a(b-a) \right] \right] \\ &= \pi R^2 \rho v_*^< (b^2 - a^2) \left[\frac{1}{\kappa^<} \ln(b-a) \left(\frac{Rv_*^<}{v} \right) + \lambda^< - \frac{1}{2\kappa^<} - \frac{1}{\kappa^<} \frac{a}{b+a} \right] \end{aligned}$$

The second integral in the expression for the mass flow rate is then:

$$2\pi\rho v_*^> \int_{bR}^R \left[\frac{1}{\kappa^>} \ln \left(\frac{yv_*^>}{v} \right) + \lambda^> \right] r dr$$

$$\begin{aligned}
&= 2\pi\rho v_*^{\lambda^>} \int_0^{R(1-b)} \left[\frac{1}{\kappa^>} \ln\left(\frac{yv_*^{\lambda^>}}{v}\right) + \lambda^> \right] (R-y) dy \\
&= 2\pi\rho v_*^{\lambda^>} \left(\frac{v}{v_*^{\lambda^>}}\right)^2 \int_0^{(1-b)(Rv_*^{\lambda^>}/v)} \left(\frac{1}{\kappa^>} \ln x + \lambda^>\right) \left(\frac{Rv_*^{\lambda^>}}{v} - x\right) dx \\
&= 2\pi\rho v_*^{\lambda^>} \left(\frac{v}{v_*^{\lambda^>}}\right)^2 \left[\frac{1}{\kappa^>} \frac{Rv_*^{\lambda^>}}{v} (x \ln x - x) - \frac{1}{\kappa^>} \left(\frac{x^2}{2} \ln x - \frac{x^2}{4}\right) \right. \\
&\quad \left. + \lambda^> \left(\frac{Rv_*^{\lambda^>}}{v}\right) x - \lambda^> \frac{1}{2} x^2 \right]_0^{(1-b)(Rv_*^{\lambda^>}/v)} \\
&= \pi R^2 \rho v_*^{\lambda^>} \left[\frac{2}{\kappa^>} \left[(1-b) \ln(1-b) \left(\frac{Rv_*^{\lambda^>}}{v}\right) - (1-b) \right] \right. \\
&\quad \left. - \frac{1}{\kappa^>} \left[(1-b)^2 \ln(1-b) \left(\frac{Rv_*^{\lambda^>}}{v}\right) - \frac{1}{2} (1-b) \right] \right. \\
&\quad \left. + 2\lambda^> (1-b) - \lambda^> (1-b)^2 \right] \\
&= \pi R^2 \rho v_*^{\lambda^>} (1-b^2) \left[\frac{1}{\kappa^>} \ln(1-b) \left(\frac{Rv_*^{\lambda^>}}{v}\right) + \lambda^> - \frac{1}{\kappa^>} \left(\frac{1}{2} + \frac{1}{1+b}\right) \right]
\end{aligned}$$

We now combine the two integrals and use the result in (b) to eliminate $\lambda^<$ in favor of $\lambda^>$; furthermore, we use the result in (a) to introduce v_{**} . This leads to:

$$w = \pi R^2 \rho v_{**} \left\{ A \left[\frac{1}{\kappa^>} \ln\left(\frac{Rv_{**}}{v} (1-b) \sqrt{1-b^2}\right) + \lambda^> \right] - B \right\}$$

in which

$$A = a^{-1/2} (b^2 - a^2)^{3/2} + (1-b^2)^{3/2}$$

$$B = a^{-1/2} (b^2 - a^2)^{3/2} \frac{1}{\kappa^<} \left(\frac{1}{2} + \frac{a}{b+a}\right) + (1-b^2)^{3/2} \frac{1}{\kappa^>} \left(\frac{1}{2} + \frac{1}{1+b}\right)$$

The expression for A does not agree with that of Meter and Bird, and we conclude that their expression is incorrect.

5C.3 Instability in a simple mechanical system

a. The centrifugal force acting on the mass is $m\Omega^2 r = m\Omega^2 L \sin \theta$. The gravitational force acting downward is mg . These forces must have a resultant in the direction of the rod, and therefore

$$\tan \theta = \frac{m\Omega^2 L \sin \theta}{mg} \quad \text{or} \quad \cos \theta = \frac{g}{\Omega^2 L}$$

If the angular velocity goes to zero, it would appear that $\cos \theta$ would go to infinity!

b. However, this formula describes the relation among the various quantities when the system is rotating and $\theta \geq 0$. When as Ω decreases, the right side of the equation attains a value of 1 when $\Omega = \Omega_{\text{thr}}$, and then $\cos \theta = 1$ and θ is zero. For for value of Ω less than its "threshold value", the value of θ remains at 0.

When one starts up the system from rest, θ will always be zero. However, if $\Omega > \Omega_{\text{thr}}$ and there is any disturbance on the system, then the system will move up to the stable curve (given by the equation in part (a)). It has to be understood that the graph we have given is only for the steady state, and that to understand the system fully, it is necessary to examine the full unsteady-state equation.

c. According to p. 12 of L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, 1990) the Lagrangian for this problem is

$$L = \frac{1}{2} mL^2 (\dot{\theta}^2 + \Omega^2 \sin^2 \theta) + mgL \cos \theta$$

Then Lagrange's equation of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

gives for the system we are considering the following equation of motion

$$mL \frac{d^2 \theta}{dt^2} = m\Omega^2 L \sin \theta \cos \theta - mg \sin \theta$$

d. Consider the lower (unstable) branch in the diagram. For very small perturbations (θ_1) to the steady state (θ_0), we have then $\sin \theta = \sin \theta_1 \approx \theta_1$ and $\cos \theta \approx 1$. Then the equation of motion becomes

$$\frac{d^2 \theta_1}{dt^2} = \left(\Omega^2 - \frac{g}{L} \right) \theta_1$$

Now we try a small perturbation of the form $\theta_1 = A \Re\{e^{-i\omega t}\}$. When we substitute this function into the differential equation we get $\omega_{\pm} = \pm i \sqrt{\Omega^2 - (g/L)}$. If $\Omega^2 < g/L$, then both roots ω_{\pm} are real and θ_1 oscillates. If $\Omega^2 > g/L$, ω_{\pm} is positive imaginary and $e^{-i\omega t}$ will increase indefinitely with time. Hence the branch $\theta = 0$ is unstable with respect to infinitesimal disturbances.

e. Next we consider the upper branch for which $\cos \theta_0 = g/\Omega^2 L$ and $\sin \theta_0 = \sqrt{1 - (g/\Omega^2 L)^2}$. Then the equation of motion becomes

$$\begin{aligned} mL \frac{d^2 \theta_1}{dt^2} &= m\Omega^2 L (\sin \theta_0 + \theta_1 \cos \theta_0) (\cos \theta_0 - \theta_1 \sin \theta_0) \\ &\quad - mg (\sin \theta_0 + \theta_1 \cos \theta_0) \\ &\approx -m\Omega^2 L \theta_1 \sin^2 \theta_0 \end{aligned}$$

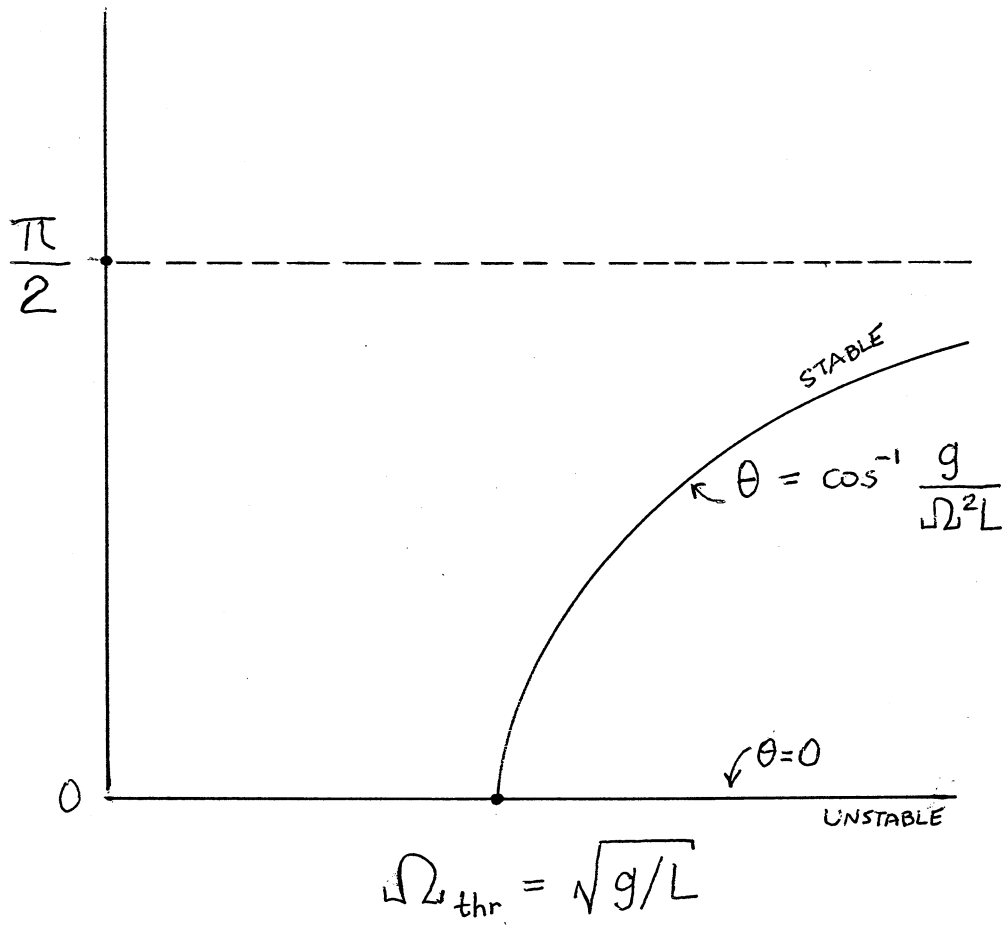
where we have neglected terms quadratic in θ_1 . Hence the equation of motion becomes

$$\frac{d^2 \theta_1}{dt^2} + \frac{1}{\Omega^2} (\Omega^2 + (g/L)) (\Omega^2 - (g/L)) \theta_1$$

We now try a solution of the form $\theta_1 = A \Re\{e^{-i\omega t}\}$ and get

$$\omega_{\pm} = \pm \frac{1}{\Omega} \sqrt{(\Omega^2 + (g/L)) (\Omega^2 - (g/L))}$$

For the upper branch, $\Omega^2 > (g/L)$, and hence both quantities ω_{\pm} are real. Hence the system is stable to small perturbations.



5D.1 Derivation of the equations of change for the Reynolds stress

Multiplication of the i th component of Eq. 5.25 by v'_j and time-smoothing gives for constant ρ gives (in the Cartesian tensor notation of §A.9, with the Einstein summation convention and with the shorthand notation $\partial_t \equiv \partial/\partial t$):

$$\rho \overline{v'_i \partial_t v'_j} = \underbrace{-\overline{v'_i \partial_j p'}}_{(2)} - \rho \left(\underbrace{\overline{v'_i \partial_k \bar{v}_k v'_j}}_{(3)} + \underbrace{\overline{v'_i \partial_k v'_k \bar{v}_j}}_{(4)} + \underbrace{\overline{v'_i \partial_k v'_k v'_j}}_{(5)} \right) + \underbrace{\mu \overline{v'_i \partial_k \partial_k v'_j}}_{(6)}$$

Then we write the same equation with i and j interchanged. When the two equations are added we get, term by term:

$$(1) \quad \rho \left(\overline{v'_i \partial_t v'_j} + \overline{v'_j \partial_t v'_i} \right) = \rho \partial_t \left(\overline{v'_i v'_j} + \overline{v'_j v'_i} \right) - \rho \left(\overline{v'_j \partial_t v'_i} + \overline{v'_i \partial_t v'_j} \right)$$

The second term on the right side is the negative of the left side. The first term on the right side can also be written as $2\rho \partial_t \overline{v'_i v'_j}$. Therefore the left side is just $\rho \partial_t \overline{v'_i v'_j}$.

$$(2) \quad -\left(\overline{v'_i \partial_j p'} + \overline{v'_j \partial_i p'} \right)$$

$$(3) \quad -\rho \left(\overline{v'_i \partial_k \bar{v}_k v'_j} + \overline{v'_j \partial_k \bar{v}_k v'_i} \right) = -\rho \left(\overline{v'_i \bar{v}_k \partial_k v'_j} + \overline{v'_j \bar{v}_k \partial_k v'_i} \right) \text{ (used 5.2-10)} \\ = -\rho \left(\overline{\bar{v}_k \partial_k v'_j v'_i} \right)$$

$$(4) \quad -\rho \left(\overline{v'_i \partial_k v'_k \bar{v}_j} + \overline{v'_j \partial_k v'_k \bar{v}_i} \right) = -\rho \left(\overline{v'_i v'_k \partial_k \bar{v}_j} + \overline{v'_j v'_k \partial_k \bar{v}_i} \right) \text{ (used 5.2-11)}$$

$$(5) \quad -\rho \left(\overline{v'_i \partial_k v'_k v'_j} + \overline{v'_j \partial_k v'_k v'_i} \right) \\ = -\rho \left(\overline{\partial_k v'_k v'_j v'_i} - \overline{v'_j v'_k \partial_k v'_i} \right) - \rho \left(\overline{v'_i v'_j \partial_k v'_k} + \overline{v'_j v'_k \partial_k v'_i} \right)$$

Note that the third term is zero by Eq. 5.2-11, and that the second and fourth terms cancel giving $-\rho \left(\overline{\partial_k v'_k v'_j v'_i} \right)$

$$(6) \quad +\mu \left(\overline{v'_i \partial_k \partial_k v'_j} + \overline{v'_j \partial_k \partial_k v'_i} \right)$$

Combining the above gives:

$$\rho \partial_t \overline{v'_i v'_j} = -(\overline{v'_i \partial_j p' + v'_j \partial_i p'}) - \rho (\overline{\bar{v}_k \partial_k v'_j v'_i}) - \rho (\overline{v'_i v'_k \partial_k \bar{v}_j} + \overline{v'_j v'_k \partial_k \bar{v}_i}) - \rho (\overline{\partial_k v'_k v'_j v'_i}) + \mu (\overline{v'_i \partial_k \partial_k v'_j} + \overline{v'_j \partial_k \partial_k v'_i})$$

or

$$\rho \partial_t \overline{v'_i v'_j} + \rho (\overline{\bar{v}_k \partial_k v'_j v'_i}) = -\rho (\overline{v'_i v'_k \partial_k \bar{v}_j} + \overline{v'_j v'_k \partial_k \bar{v}_i}) - \rho (\overline{\partial_k v'_k v'_j v'_i}) - (\overline{v'_i \partial_j p' + v'_j \partial_i p'}) + \mu (\overline{v'_i \partial_k \partial_k v'_j} + \overline{v'_j \partial_k \partial_k v'_i})$$

The two terms on the left side are the substantial derivative term in Eq. 5D.1-1. The remainder of the terms are set out in the same order as in Eq. 5D.1-1.

5D.2 Kinetic energy of turbulence

Taking the trace of Eq. 5D.1-1, we get

$$\rho \frac{D}{Dt} \overline{(\mathbf{v}' \cdot \mathbf{v}')} = -2\rho \overline{(\mathbf{v}' \mathbf{v}' : \nabla \bar{\mathbf{v}})} - \rho \overline{(\nabla \cdot (\mathbf{v}' \cdot \mathbf{v}') \mathbf{v}')} \\ - 2 \overline{(\mathbf{v}' \cdot \nabla p')} + 2\mu \overline{(\mathbf{v}' \cdot \nabla^2 \mathbf{v}')}$$

Modify the third term on the right as follows:

$$-2 \overline{(\mathbf{v}' \cdot \nabla p')} = -2 \overline{(\nabla \cdot p' \mathbf{v}')} + 2 \overline{(p' (\nabla \cdot \mathbf{v}'))}$$

The last term on the right is zero according to Eq. 5.2-11. Then divide the first equation above by 2 to get

$$\frac{D}{Dt} \overline{\frac{1}{2} \rho v'^2} = -\rho \overline{(\mathbf{v}' \mathbf{v}' : \nabla \bar{\mathbf{v}})} - \overline{(\nabla \cdot (\frac{1}{2} \rho v'^2) \mathbf{v}')} \\ - \overline{(\nabla \cdot p' \mathbf{v}')} + \mu \overline{(\mathbf{v}' \cdot \nabla^2 \mathbf{v}')}$$

which is Eq. 5D.2-1.

For an interpretation of this equation, see pp. 63 et seq. in the book by Tennekes and Lumley cited on p. 176.

6A.1 Pressure drop required for a pipe with fittings.

The average velocity at the given conditions is

$$\langle v \rangle = \frac{4(w/\rho)}{\pi D^2} = \frac{(4)(197 \text{ m}^3/\text{s})}{(\pi)(0.25 \text{ m})^2} = 40.1 \text{ m/s}$$

and the Reynolds number is

$$\frac{D\langle v \rangle}{\nu} = (0.25 \text{ m})(40.1 \text{ m/s}) / (1.0037 \times 10^{-6} \text{ m}^2/\text{s}) = 9.99 \times 10^6$$

Thus, the flow is turbulent, and Fig. 6.2-1 gives $f=0.0020$ for hydraulically smooth pipe. The total equivalent length of the pipe and fittings is

$$\begin{aligned} L_e &= 1234 \text{ m of pipe} \\ &+ (4)(32)(0.25) \text{ m equivalent for 4 } 90^\circ \text{ elbows} \\ &+ (2)(15)(0.25) \text{ m equivalent for 2 } 90^\circ \text{ elbows} \\ &= 1274.5 \text{ m} \end{aligned}$$

The required pressure drop, according to Eq. 6.1-4 with L replaced by L_e , is then

$$\begin{aligned} (p_0 - p_L) &= 2 \frac{L_e}{D} \cdot \rho \langle v \rangle^2 f \\ &= 2 \frac{1273.5 \text{ m}}{0.25 \text{ m}} (998 \text{ kg/m}^3) (40.1 \text{ m/s})^2 (0.0020) \\ &= 3.3 \times 10^7 \text{ Pa} = 4.7 \times 10^3 \text{ psi} \end{aligned}$$

6A.2 Pressure difference required for flow in pipe with elevation change.

In this problem the pipe diameter is $(3.068 \text{ in.})/(39.37 \text{ in./m}) = 0.07793 \text{ m}$, the mass flow rate is

$$w = (18/60 \text{ gal/s})(3.7853 \text{ lit/gal})(0.9982 \text{ kg/lit}) = 1.13 \text{ kg/s},$$

the average velocity is

$$\langle v \rangle = \frac{4w}{\pi D^2 \rho} = \frac{(4)(1.13 \text{ kg/s})}{\pi(0.07793 \text{ m})^2(998.2 \text{ kg/m}^3)} = 0.237 \text{ m/s},$$

and the Reynolds number is

$$\text{Re} = \frac{D\langle v \rangle \rho}{\mu} = \frac{(0.07793 \text{ m})(0.237 \text{ m/s})(998.2 \text{ kg/m}^3)}{(1.002 \times 10^{-3} \text{ kg/m}\cdot\text{s})} = 1.84 \times 10^4$$

From Fig. 6.2-2 we find that for this Re value, $f = 0.0066$ for smooth tubes. Hence,

$$\begin{aligned} p_0 - p_L &= -\rho g(h_0 - h_L) + 2\frac{L_e}{D} \cdot \rho \langle v \rangle^2 f \\ &= -(998.2 \text{ kg/m}^3)(9.807 \text{ m/s}^2)((-50 \times 0.3048/\sqrt{2}) \text{ m}) \\ &\quad + 2[(50 \times 12/3.068) + (2 \times 15)] \cdot (998.2 \text{ kg/m}^3)(0.237 \text{ m/s})^2(0.0066) \\ &= 1.055 \times 10^5 \text{ Pa} + 167 \text{ Pa} \\ &= 1.055 \times 10^5 \text{ Pa} = 15.3 \text{ psi} \end{aligned}$$

The corresponding calculation in terms of lb_m , ft and s, neglecting the small friction term, is

$$\begin{aligned} p_0 - p_L &= (62.4 \times 0.9982 \text{ lb}_m/\text{ft}^3)(32.174 \text{ ft/s}^2)((50/\sqrt{2}) \text{ ft}) \\ &= 7.09 \times 10^4 \text{ lb}_m/\text{ft}\cdot\text{s}^2 = 15.3 \text{ psia} \end{aligned}$$

6A.3 Flow rate for a given pressure drop.

The quantities needed for this calculation are as follows, in units of lb_m , ft, and s:

$$\begin{aligned}p_0 - p_L &= (0.25 \text{ lb}_f/\text{in}^2)(144 \text{ in}^2/\text{ft}^2)(32.174 \text{ lb}_m\text{ft}/\text{s}^2 \cdot \text{lb}_f) \\ &= 1.16 \times 10^3 \text{ lb}_m/\text{ft} \cdot \text{s}^2; \\ D &= 0.5 \text{ ft}; \quad \rho = 62.4 \text{ lb}_m/\text{ft}^3; \\ L &= 1320 \text{ ft}; \quad \mu = 6.73 \times 10^{-4} \text{ lb}_m/\text{ft} \cdot \text{s}\end{aligned}$$

Hence, the solution must lie on the locus

$$\begin{aligned}\text{Re}\sqrt{f} &= \frac{D\rho}{\mu} \sqrt{\frac{(p_0 - p_L)D}{2L\rho}} \\ &= \frac{(0.5)(62.4)}{6.73 \times 10^{-4}} \sqrt{\frac{(1.15 \times 10^{-3})(0.5)}{2(1320)(62.4)}} \\ &= (4.64 \times 10^4) \sqrt{3.52 \times 10^{-3}} = 2.74 \times 10^3\end{aligned}$$

Method B: The last equation gives a straight line on the logarithmic plot of f vs. Re , passing through $f = 1$ at $\text{Re} = 2.74 \times 10^3$ and through $f = 0.01$ at $\text{Re} = 2.74 \times 10^4$, and intersecting the f curve for smooth tubes at $\text{Re} = 3.6 \times 10^4$. Hence, the average velocity is

$$\langle v \rangle = \frac{\text{Re}}{(D\rho/\mu)} = \frac{3.6 \times 10^4}{4.64 \times 10^4} = 0.78 \text{ ft/s}$$

and the volume rate of flow is

$$\begin{aligned}Q &= \frac{\pi D^2}{4} \langle v \rangle \\ &= \frac{(3.1416)(0.5^2)}{4} (0.78) = 0.152 \text{ ft}^3/\text{s} \\ &= 68 \text{ U.S. gal/hr}\end{aligned}$$

Method A gives the same result if the plot of f vs. Re is accurately drawn

6A.4 Motion of a sphere in a liquid.

The force of gravity on the sphere is

$$F_{\text{grav}} = mg = (0.0500 \text{ g})(980.665 \text{ cm/s}^2) = 49.0_3 \text{ dynes}$$

The buoyant force of the fluid on the sphere is

$$\begin{aligned} F_{\text{buoy}} &= (4/3)\pi R^3 \rho g \\ &= (4/3)\pi(0.25 \text{ cm})^3(0.900 \text{ g/cm}^3)(980.665 \text{ cm/s}^2) = 57.7_7 \text{ dynes} \end{aligned}$$

a. The resultant upward force is

$$F_{\text{buoy}} - F_{\text{grav}} = 57.7_7 - 49.0_3 = 8.7_4 \text{ dynes}$$

and is balanced, at steady state, by an equal and opposite drag force $F_k = 8.7_4$ dynes.

b. The friction factor is defined by

$$F_k = (\pi R^2)\left(\frac{1}{2}\right)\rho v_{\infty}^2 f$$

Thus, for this system,

$$\begin{aligned} f &= \frac{F_k}{\left(\frac{\pi D^2}{4}\right)\left(\frac{1}{2}\rho v_{\infty}^2\right)} = \frac{8F_k}{\pi D^2 \rho v_{\infty}^2} \\ &= \frac{8(8.7_4 \text{ dynes})}{\pi(0.500 \text{ cm})^2(0.900 \text{ g/cm}^3)(0.500 \text{ cm/s})^2} \\ &= 3.9_6 \times 10^2 \end{aligned}$$

c. From Fig. 6.3-1 we see that f is very close to its creeping-flow asymptote, $24/\text{Re}$. To the same approximation,

$$\frac{Dv_{\infty}\rho}{\mu} \approx 24/f = 0.061$$

Hence,

$$\begin{aligned} \mu &= \frac{Dv_{\infty}\rho f}{24} = \frac{(0.5 \text{ cm})(0.5 \text{ cm/s})(0.9 \text{ g/cm}^3)(3.9_6 \times 10^2)}{24} \\ &= 3.7 \text{ g/cm}\cdot\text{s} = 3.7 \times 10^2 \text{ cp} \end{aligned}$$

6A.5 Sphere diameter for a given terminal velocity.

a. Method A: Replot the f -curve of Fig. 6.3-1 as f/Re (which does not contain D) vs. Re . Then from this curve we can find the value of Re for any calculated value of f/Re , and determine D as $\text{Re}\mu/\rho v_\infty$.

a. Method B: On the log-log plot of $f = f(\text{Re})$, plot also the locus $f = (f/\text{Re})\text{Re}$, which will be a line of slope 1, and find the desired Re at the intersection of the two loci. This method avoids any need to prepare an auxiliary plot.

b. The data of Problem 2A.4 give

$$\begin{aligned}v_\infty &= (1 \text{ ft/s})(12 \times 2.54 \text{ cm/ft}) = 30.48 \text{ cm/s} \\ \rho &= 0.045 \text{ lb}_m/\text{ft}^3 (453.59 \text{ g/lb}_m)(12 \times 2.54 \text{ cm/ft})^{-3} \\ &= 7.2 \times 10^{-4} \text{ g/cm}^{-3} \\ \rho_s &= 1.2 \text{ g/cm}^3 \\ \mu &= 2.6 \times 10^{-4} \text{ g/cm}\cdot\text{s} \\ g &= 980.7 \text{ cm/s}^2\end{aligned}$$

from which we calculate

$$\begin{aligned}f/\text{Re} &= \frac{4}{3} \frac{g\mu}{\rho v_\infty^2} \left(\frac{\rho_s - \rho}{\rho} \right) \\ &= \frac{4(980.7)(0.00026)}{3 \cdot 1.2 v_\infty^3} \left(\frac{1.2 - 7.2 \times 10^{-4}}{7.2 \times 10^{-4}} \right) \\ &= 27.8\end{aligned}$$

We therefore draw a line of slope 1 through $f = 27.8$, $\text{Re} = 1$ on Fig. 6.3-1. This line intersects the f vs. Re curve at $\text{Re} = 0.95$. The particle diameter is then calculated as

$$D = \frac{\text{Re}\mu}{v_\infty\rho} = \frac{(0.95)(2.6 \times 10^{-4})}{(30.48)(7.2 \times 10^{-4})} = (0.0112 \text{ cm})(10^4 \text{ microns/cm}) = 112 \text{ microns}$$

c. Here v_∞ is 10 times larger, giving $f/\text{Re} = 27.8 \times 10^{-3}$. This locus intersects the $f(\text{Re})$ curve at $\text{Re} = 75$. Hence, at this gas velocity the diameter of the largest particle that can be lost is

$$D = \frac{(75)(2.6 \times 10^{-4})}{(30.48)(7.2 \times 10^{-4})} = (0.89 \text{ cm})(10^4 \text{ microns/cm}) = 0.89 \times 10^4 \text{ microns}$$

6A.6 Estimation of void fraction of a packed column.

The superficial velocity is

$$v_0 = \frac{(244 \text{ lb/min})(1 \text{ min}/60 \text{ s})(453.59 \text{ g/lb})}{(1.2865 \text{ g/cm}^3)(146 \text{ in}^2)(2.54 \text{ cm/in})^2} = 1.522 \text{ cm/s}$$

According to Eq. 6.4-9 (the Blake-Kozeny equation, developed for laminar flows),

$$\begin{aligned} \frac{\varepsilon^3}{(1 - \varepsilon)^2} &= \frac{150\mu Lv_0}{D_p^2 \Delta p} \\ &= \frac{150(0.565 \text{ g/cm}\cdot\text{s})(73 \times 2.54 \text{ cm})(1.522 \text{ cm/s})}{(0.2 \text{ cm})^2(158 \text{ psi})(68947 \text{ dynes/cm}^2\cdot\text{psi})} \\ &= 0.0549 \end{aligned}$$

Solving this equation for ε , we find $\varepsilon = 0.30$; hence,

$$\frac{D_p v_0 \rho}{\mu} \cdot \frac{1}{1 - \varepsilon} = \frac{(0.2 \text{ cm})(1.522 \text{ cm/s})(1.287 \text{ g/cm}^3)}{(0.565 \text{ g/cm}\cdot\text{s})} \frac{1}{(1 - 0.30)} = 1.00$$

which indicates (see Fig. 6.4-2) that it was appropriate to use the Blake-Kozeny equation here.

6A.7 Estimation of pressure drops in annular flow.

a. The definition of f in Eq. 6.1-1 gives

$$f = \frac{F_k}{2\pi RL(1 + \kappa)\rho\langle\bar{v}_z\rangle^2/2} = \frac{\pi R^2(1 - \kappa^2)(\mathcal{P}_0 - \mathcal{P}_L)}{2\pi RL(1 + \kappa)\rho\langle\bar{v}_z\rangle^2/2} = \frac{R(1 - \kappa)(\mathcal{P}_0 - \mathcal{P}_L)}{L\rho\langle\bar{v}_z\rangle^2}$$

Equations 6A.7-1 and 3 give

$$K = \frac{16}{f} \frac{\mu}{D(1 - \kappa)\langle\bar{v}_z\rangle\rho}$$

Insertion of the previous result for f gives

$$K = \frac{16L\rho\langle\bar{v}_z\rangle^2}{R(1 - \kappa)(\mathcal{P}_0 - \mathcal{P}_L)} \cdot \frac{\mu}{D(1 - \kappa)\langle\bar{v}_z\rangle\rho} = \frac{8L\mu\langle\bar{v}_z\rangle}{R^2(1 - \kappa)^2(\mathcal{P}_0 - \mathcal{P}_L)}$$

Equation 2.4-16 gives

$$\frac{\langle\bar{v}_z\rangle}{(\mathcal{P}_0 - \mathcal{P}_L)} = \frac{R^2}{8\mu L} \left[\frac{1 - \kappa^4}{1 - \kappa^2} - \frac{1 - \kappa^2}{\ln(1/\kappa)} \right]$$

Combining the last two results, we get the relation

$$K = \frac{1}{1 - \kappa^2} \left[\frac{1 - \kappa^4}{1 - \kappa^2} - \frac{1 - \kappa^2}{\ln(1/\kappa)} \right]$$

needed to make Eqs. 6A.7-1 and 3 consistent with Eq. 2.4-16 for laminar flow. The K values given by this formula are in excellent agreement with those tabulated on page 194 and recommended for turbulent flow as well.

b. The data for this operation are

$$\mu = 1.139 \text{ cp (=mPa}\cdot\text{s)} = 7.66 \times 10^{-4} \text{ lb}_m/\text{ft}\cdot\text{s}; \quad \rho = 62.2 \text{ lb}_m/\text{ft}^3;$$

$$D = 15 \text{ in} = 1.25 \text{ ft}; \quad \kappa = 6 \text{ in}/15 \text{ in} = 0.4; \quad G = 3.801;$$

$$H = 0.131; \quad K = 0.6759; \quad w/\rho = 1500 \text{ ft}^3/\text{s};$$

$$\langle\bar{v}_z\rangle = \frac{4(w/\rho)}{\pi[D^2 - (\kappa D)^2]} = \frac{4(1600 \text{ ft}^3/\text{s})}{\pi[1.25^2 - 0.5^2 \text{ ft}^2]} = 388 \text{ ft/s};$$

$$\begin{aligned} \text{Re}_\kappa &= K \frac{D(1 - \kappa)\langle\bar{v}_z\rangle\rho}{\mu} \\ &= 0.6759 \frac{(1.25 \text{ ft})(1 - 0.4)(388 \text{ ft/s})(62.3 \text{ lb}_m/\text{ft}^3)}{(7.66 \times 10^{-4})} = 1.6 \times 10^7 \end{aligned}$$

Equation 6A.7-2 then gives:

$$\frac{1}{\sqrt{f}} = 3.801 \log_{10}(1.6 \times 10^7 \sqrt{f}) - 0.131$$

Solving by iteration, we get $f = 0.00204$. The longitudinal pressure gradient in a horizontal flow is then

$$\begin{aligned} \frac{(p_0 - p_L)}{L} &= \frac{2f\rho\langle\bar{v}_z\rangle^2}{D(1 - \kappa)} \\ &= \frac{2(0.00204)(62.3 \text{ lb}_m/\text{ft}^3)(388 \text{ ft/s})^2}{(1.25 \text{ ft})(1 - 0.4)} \\ &= 5.06 \times 10^4 \text{ poundals/ft}^3 = 10.9 \text{ psi/ft} \end{aligned}$$

c. The mean hydraulic radius for this system is

$$R_h = \frac{S}{Z} = \frac{\pi D^2(1 - \kappa^2)/4}{\pi D(1 - \kappa)} = \frac{D(1 - \kappa)}{4}$$

and the corresponding Reynolds number is

$$\text{Re}_h = \frac{4R_h\langle\bar{v}_z\rangle\rho}{\mu} = \frac{D(1 - \kappa)\langle\bar{v}_z\rangle\rho}{\mu} = \text{Re}/K = 2.37 \times 10^7$$

Equation 6.2-15 then gives $f = 0.00181$, which is 0.885 times the value found in part (b). The predicted pressure gradient is reduced correspondingly, giving

$$\frac{(p_0 - p_L)}{L} = 4.48 \text{ poundals/ft}^3 = 9.7 \text{ psi/ft}$$

6A.8 Force on a water tower in a gale.

The data for this problem are:

$$\begin{aligned} D &= 40 \text{ ft for spherical tank;} & v_{\infty} &= (100 \text{ mi/hr})(5280/3600) = 147 \text{ ft/s;} \\ \rho &= 0.08 \text{ lb}_m/\text{ft}^3; & \mu &= 0.017 \text{ cp} = 1.14 \times 10^{-5} \text{ lb}_m/\text{ft}\cdot\text{s} \end{aligned}$$

The Reynolds number for the tank is

$$\text{Re} = \frac{Dv_{\infty}\rho}{\mu} = \frac{(40 \text{ ft})(147 \text{ ft/s})(0.08 \text{ lb}_m/\text{ft}^3)}{(1.14 \times 10^{-5} \text{ lb}_m/\text{ft}\cdot\text{s})} = 4.1 \times 10^7$$

The friction factor is approximated as 0.5 by extrapolation of Fig. 6.3-1.

The horizontal force of the wind on the tank is then

$$\begin{aligned} F_k &= \left(\frac{\pi D^2}{4}\right)\left(\frac{1}{2}\rho v_{\infty}^2\right)f \\ &= \left(\frac{\pi(40 \text{ ft})^2}{4}\right)\left(\frac{1}{2}(0.08 \text{ lb}_m/\text{ft}^3)(147 \text{ ft/s})^2\right)(0.5) \\ &= 5.4 \times 10^5 \text{ poundals} = 1.7 \times 10^4 \text{ lb}_f = 7.5 \times 10^4 \text{ Newtons} \end{aligned}$$

6A.9 Flow of gas through a packed column.

For this compressible-flow problem, we write Eq. 6.4-12 in differential form with $v_0 = G_0/\rho$:

$$-\frac{dp}{dz} = 150 \frac{\mu G_0 (1-\varepsilon)^2}{\rho D_p^2 \varepsilon^3} + \frac{7 G_0^2 (1-\varepsilon)}{4 \rho D_p \varepsilon^3}$$

Inserting the ideal gas formula

$$\rho = \frac{pM}{RT}$$

and integrating from $z = 0$ to $z = L$, we get the following implicit expression for the superficial mass flux G_0 :

$$\left[150 \frac{\mu G_0 (1-\varepsilon)^2}{D_p^2 \varepsilon^3} + \frac{7 G_0^2 (1-\varepsilon)}{4 D_p \varepsilon^3} \right] L = \frac{M}{2RT} (p_0^2 - p_L^2)$$

The terms are then calculated for this system in cgs units:

$$\begin{aligned} & 150 \frac{\mu G_0 (1-\varepsilon)^2}{D_p^2 \varepsilon^3} L \\ = & 150 \frac{(1.495 \times 10^{-4} \text{ g/cm}\cdot\text{s})(G_0 \text{ g/cm}^2\cdot\text{s})(1-0.41)^2}{(2.54/16 \text{ cm})^2 (0.41)^3} (5.5 \times 30.48 \text{ cm}) \\ & = (753.4 G_0) \text{ g}^2/\text{cm}^4\cdot\text{s}^2 \\ & \frac{7 G_0^2 (1-\varepsilon)}{4 D_p \varepsilon^3} L \\ & = \frac{7 (G_0 \text{ g/cm}^2\cdot\text{s})^2 (1-0.41)}{4 (2.54/16 \text{ cm}) (0.41)^3} (5.5 \times 30.48 \text{ cm}) \\ & = (15820 G_0^2) \text{ g}^2/\text{cm}^4\cdot\text{s}^2 \\ \frac{M}{2RT} (p_0^2 - p_L^2) & = \frac{44.01 \text{ g/g}\cdot\text{mol}}{(8.31451 \times 10^7 \times 300 \text{ g}\cdot\text{cm}^2/\text{s}^2\cdot\text{g}\cdot\text{mol})} \\ & \times [(25 \times 1.0133 \times 10^6)^2 - (3 \times 1.0133 \times 10^6)^2] \text{ g}^2/\text{cm}^2\cdot\text{s}^4 \\ & = (1.116 \times 10^6) \text{ g}^2/\text{cm}^4\cdot\text{s}^4 \end{aligned}$$

Combining these results, we get the following quadratic equation for G_0 ,

$$15820 G_0^2 + 753.4 G_0 = 1.116 \times 10^6$$

which has the roots

$$\begin{aligned} G_0 & = \frac{-753.46 \pm \sqrt{753.46^2 + 4(15820)(1.116 \times 10^6)}}{2 \times 15820} \\ & = \frac{-753.46 \pm 265746.2}{2 \times 15820} \end{aligned}$$

The positive root is 8.375 g/cm²/s. the mass flow rate is then

$$w = (\pi/4) D^2 G_0 = (0.7854)(4 \times 2.54 \text{ cm})^2 (8.375 \text{ g/cm}^2/\text{s}) = 679 \text{ g/s}$$

An identical result for w is obtained if μ is assumed constant at the value $\bar{\mu}$ —

6B.1 Effect of error in friction factor calculations

The Blasius formula for the friction factor for turbulent flow in circular tubes is Eq. 6.2-12:

$$f = \frac{0.0791}{\text{Re}^{1/4}}$$

We now form the differential of f , thus

$$df = -\frac{1}{4} \frac{0.0791}{\text{Re}^{5/4}} d\text{Re}$$

and regard the differentials as the errors in the relevant quantities. This result may be rearranged thus

$$df = -\frac{0.0791}{\text{Re}^{1/4}} \frac{1}{4\text{Re}} d\text{Re} = -f \cdot \frac{1}{4\text{Re}} d\text{Re}$$

From this we find

$$\frac{df}{f} = -\frac{1}{4} \frac{d\text{Re}}{\text{Re}}$$

The quantity df/f represents the fractional error in f , and $d\text{Re}/\text{Re}$ is the fractional error in Re .

Therefore, if the Reynolds number is too low by 4%, then the friction factor will be too high by 1%.

6B.2 Friction factor for flow along a flat plate

a. Equation 4.4-30 gives the kinetic force acting on both sides of the flat plate for *laminar flow*:

$$F_k = 1.328\sqrt{\rho\mu LW^2v_\infty^3}$$

and an appropriate Reynolds number for a plate of length L is

$$\text{Re}_L = \frac{Lv_\infty\rho}{\mu}$$

Then, using the definition of the friction factor in Eq. 6.1-1, we write

$$f = \frac{F_k}{AK} = \frac{1.328\sqrt{\rho\mu LW^2v_\infty^3}}{2WL \cdot \frac{1}{2}\rho v_\infty^2} = 1.328\sqrt{\frac{\mu}{Lv_\infty\rho}} = \frac{1.328}{\sqrt{\text{Re}_L}}$$

b. For *turbulent flow*, the force acting on the plate is given by Eq. 6B.2-1 as

$$F_k = 0.074\rho v_\infty^2 WL (Lv_\infty\rho/\mu)^{-1/5}$$

Then the friction factor is

$$f = \frac{F_k}{AK} = \frac{0.074\rho v_\infty^2 WL (Lv_\infty\rho/\mu)^{-1/5}}{2WL \cdot \frac{1}{2}\rho v_\infty^2} = \frac{0.074}{\text{Re}_L^{1/5}}$$

Here we have used the same definition of Reynolds number as in part (a).

6B.3 Friction factor for laminar flow in a slit

a. The mass flow rate through a slit of width W , thickness $2B$, and length L is, for laminar flow

$$w = \frac{2(\mathcal{P}_0 - \mathcal{P}_L)B^3W\rho}{3\mu L} = \rho\langle v_z \rangle(2BW)$$

and the kinetic force acting on the walls by the fluid is

$$F_k = (\mathcal{P}_0 - \mathcal{P}_L)(2BW)$$

Then the friction factor is

$$f = \frac{F_k}{AK} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)(2BW)}{(2WL)\left(\frac{1}{2}\rho\langle v_z \rangle^2\right)}$$

We now replace one of the $\langle v_z \rangle$ in the denominator of this expression by using the above expression for the mass flow rate. This gives

$$f = \frac{(\mathcal{P}_0 - \mathcal{P}_L)(2BW)}{(2WL)\left(\frac{1}{2}\rho\langle v_z \rangle\right)} \cdot \frac{3\mu L}{(\mathcal{P}_0 - \mathcal{P}_L)B^2} = \frac{12}{2B\langle v_z \rangle\rho/\mu} = \frac{12}{\text{Re}}$$

b Next we try using the mean hydraulic radius empiricism suggested on p. 183. For the plane slit, R_h is

$$R_h = \frac{S}{Z} = \frac{2BW}{2W + 4B} \approx B$$

In the last step we have made the assumption (consistent with the derivation of the velocity profile for the slit) that $B \ll W$. We now replace the D in the tube flow friction factor by $4R_h$. Then

$$f = \frac{16}{D\langle v_z \rangle\rho/\mu} \rightarrow \frac{16}{4R_h\langle v_z \rangle\rho/\mu} = \frac{16}{4B\langle v_z \rangle\rho/\mu} = \frac{8}{2B\langle v_z \rangle\rho/\mu} = \frac{8}{\text{Re}}$$

This poor result emphasizes the statement made just after Eq. 6.2-18.

6B.4 Friction factor for a rotating disk

a. Laminar flow

$$f = \frac{T_z}{AKR} = \frac{0.616\pi\rho R^4 \sqrt{\mu\Omega^3/\rho}}{(2\pi R^2)\left(\frac{1}{2}\rho\Omega^2 R^2\right)R} = 0.616 \sqrt{\frac{\mu}{R^2\Omega\rho}} = \frac{0.616}{\text{Re}^{1/2}}$$

b. Turbulent flow

$$f = \frac{T_z}{AKR} = \frac{0.073\rho\Omega^2 R^5 (\mu/R^2\Omega\rho)^{1/5}}{(2\pi R^2)\left(\frac{1}{2}\rho\Omega^2 R^2\right)R} = 0.073 \left(\frac{\mu}{R^2\Omega\rho}\right)^{1/5} = \frac{0.073}{\text{Re}^{1/5}}$$

6B.5 Turbulent flow in horizontal pipes.

a. Equations 6.1-4 and 6.2-12 give, for horizontal flow in smooth pipe with fixed w , ρ , μ , and L ,

$$\begin{aligned}(p_0 - p_L) &= 2 \frac{L}{D} \rho \langle v \rangle^2 \cdot f \\ &= 2 \frac{L}{D} (\rho) \left(\frac{4w}{\rho \pi D^2} \right)^2 (0.0791) \left(\frac{4w}{\pi D \mu} \right)^{-1/4} \\ &\propto D^{-5+1/4}\end{aligned}$$

According to this formula, the replacement of the smooth pipe of diameter D by a smooth one of diameter $D/2$, with all other right-hand variables unchanged, will alter $(p_0 - p_L)$ by a factor of $(1/2)^{-5+1/4} \approx 27$.

b. If the Blasius expression for F is not appropriate (e.g., because the range of Re is higher or non-smooth pipes are used), then one must calculate $Re = 4w/\pi D\mu$ for the present operation, read f_{old} from Fig. 6.2-2 at that Re with the applicable roughness measure k/D , and then read f_{new} at the new values of $4w/\pi D\mu$ and k/D . Then the calculation in (a) needs to be modified only by using f_{new}/f_{old} in place of $(D_{new}/D_{old})^{1/4}$. The inverse 5th-power of D will still be dominant in the calculation of the change in $(p_0 - p_L)$.

6B.6 Inadequacy of mean hydraulic radius for laminar flow

a. The mean hydraulic radius for the annulus is (Eq. 6.2-16)

$$R_h = \frac{S}{Z} = \frac{\pi R^2 - \pi(\kappa R)^2}{2\pi R + 2\pi(\kappa R)} = \frac{R(1 - \kappa^2)}{2(1 + \kappa)} = \frac{R}{2}(1 - \kappa)$$

Then, according to Eq. 6.2-18, the Reynolds number is

$$\text{Re}_h = \frac{4R_h \langle v_z \rangle \rho}{\mu} = \frac{2R(1 - \kappa) \langle v_z \rangle \rho}{\mu}$$

Then, according to Eq. 6.2-17 the friction factor is

$$f = \left(\frac{R_h}{L} \right) \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{\frac{1}{2} \rho \langle v_z \rangle^2} \right) = \frac{16}{\text{Re}_h} = \frac{4\mu}{R_h \langle v_z \rangle \rho}$$

or

$$\langle v_z \rangle = \frac{(\mathcal{P}_0 - \mathcal{P}_L) R_h^2}{2\mu L} = \frac{(\mathcal{P}_0 - \mathcal{P}_L) R^2 (1 - \kappa)^2}{8\mu L}$$

This is the mean-hydraulic-radius analog of Eq. 2.4-16.

b. When $\kappa = \frac{1}{2}$, the above result for the mean-hydraulic-radius empiricism gives:

$$\langle v_z \rangle = \frac{(\mathcal{P}_0 - \mathcal{P}_L) R^2}{8\mu L} (0.250)$$

whereas Eq. 2.4-16 (the exact result) gives

$$\begin{aligned} \langle v_z \rangle &= \frac{(\mathcal{P}_0 - \mathcal{P}_L) R^2}{8\mu L} \left[\frac{1 - \kappa^4}{1 - \kappa^2} - \frac{1 - \kappa^2}{\ln(1/\kappa)} \right] \\ &= \frac{(\mathcal{P}_0 - \mathcal{P}_L) R^2}{8\mu L} \left[\frac{1 - (\frac{1}{2})^4}{1 - (\frac{1}{2})^2} - \frac{1 - (\frac{1}{2})^2}{\ln 2} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{8\mu L} \left[\frac{\left(\frac{15}{16}\right)}{\left(\frac{3}{4}\right)} - \frac{\left(\frac{3}{4}\right)}{0.6931} \right] \\
&= \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{8\mu L} \left[1.25 - \frac{0.750}{0.6931} \right] \\
&= \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{8\mu L} [1.25 - 1.0821] \\
&= \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{8\mu L} (0.1679)
\end{aligned}$$

The percentage error in the mean-hydraulic-radius empiricism is then

$$\frac{0.2500 - 0.1679}{0.1679} \times 100 = \frac{0.0821}{0.1679} \times 100 = 49\%$$

This confirms the inadequacy of the mean-hydraulic-radius empiricism for laminar flow.

6B.7 Falling sphere in Newton's drag-law region.

Let z be the (downward) displacement of the sphere from its initial position, and $v = v_z = dz/dt$ denote the instantaneous velocity of the sphere. Then the z -component of Newton's second law gives, for a falling sphere of mass m ,

$$m \frac{dv}{dt} = F_z$$

in which

$$\begin{aligned} F_z &= mg_z - F_{kz} \\ &= mg - (\pi R^2) \left(\frac{1}{2} \rho v^2 \right) (0.44) \\ &= mg(1 - c^2 v^2) \end{aligned}$$

with c^2 defined by the last equation. Hence,

$$\frac{dv}{dt} = g(1 - c^2 v^2)$$

and integration from the initial condition $v = 0$ at $t = 0$ gives

$$\int_0^v \frac{dv}{1 - c^2 v^2} = g \int_0^t dt$$

or

$$\frac{1}{c} \tanh^{-1} cv = gt$$

whence

$$\frac{dz}{dt} \equiv v = \frac{1}{c} \tanh cgt$$

Note that as $t \rightarrow \infty$, $\tanh cgt \rightarrow 1$. Hence, the "terminal velocity" is $1/c$. A second integration then gives (for $z = 0$ at $t = 0$):

$$\int_0^z dz = \frac{1}{c} \int_0^t \tanh cgt dt$$

or

$$\begin{aligned} z &= \frac{1}{c^2 g} \int_0^{cgt} \tanh \beta d\beta \\ &= \frac{1}{c^2 g} \ln \cosh cgt \end{aligned}$$

Now $mgc^2 = 0.22\pi R^2 \rho$, so that

$$\begin{aligned} c &= \sqrt{0.22\pi R^2 \rho / mg} \\ &= \sqrt{\frac{0.22\pi R^2 \rho}{\frac{4}{3}\pi R^2 \rho_{\text{sph}} g}} \\ &= \sqrt{\left(\frac{3}{4}\right) (0.22) \left(\frac{\rho}{\rho_{\text{sph}}}\right) \left(\frac{1}{Rg}\right)} \end{aligned}$$

Note that this solution assumes that the particle is in the Newton's drag law region during its entire trajectory. However, the initial condition that $v = 0$ at $t = 0$ is clearly outside the Newton's drag law region.

6B.8 Design of an experiment to verify the f vs Re chart for spheres

a. We start with the friction-factor expression in Eq. 6.1-7. Since we need to have a D expression containing the Reynolds number (but not the terminal velocity), we eliminate the velocity in favor of Re . This gives

$$f = \frac{4}{3} \frac{gD}{v_{\infty}^2} \left(\frac{\rho_{\text{sph}} - \rho}{\rho} \right) = \frac{4}{3} \frac{gD}{Re^2} \left(\frac{D\rho}{\mu} \right)^2 \left(\frac{\rho_{\text{sph}} - \rho}{\rho} \right) = \frac{4}{3} \frac{gD^3}{Re^2 \mu^2} (\rho_{\text{sph}} - \rho) \rho$$

We now solve this expression for D to get

$$D = \sqrt[3]{\frac{3 f Re^2 \mu^2}{4 (\rho_{\text{sph}} - \rho) \rho g}}$$

b. Next, substituting numerical values into the formula, we get

$$D = \sqrt[3]{\frac{3 (1)(100)^2 (10^{-2})^2}{4 (8-1)(1)(980.7)}} = 0.048 \text{ cm}$$

6B.9 Friction factor for flow past an infinite cylinder

We select the area A of Eq. 6.1-1 to be the area "seen" by the approaching fluid, which is DL , and K to be $\frac{1}{2}\rho v_{\infty}^2$. Then combining Eqs. 6.1-1 and 6B.9-1 gives

$$F_k = fAK = \frac{4\pi\mu v_{\infty} L}{\ln(7.4/Re)}$$

We now solve for f and insert the expressions for A and K to get

$$f = \frac{4\pi\mu v_{\infty} L}{\ln(7.4/Re)} \frac{1}{(DL)\left(\frac{1}{2}\rho v_{\infty}^2\right)} = \frac{8\pi}{\ln(7.4/Re)} \frac{1}{(Dv_{\infty}\rho/\mu)}$$

Therefore

$$f = \frac{8\pi}{Re \ln(7.4/Re)}$$

6C.1 Two-dimensional particle directories

a. We start with Newton's law of motion for a particle

$$\frac{d}{dt} m\mathbf{v} = \sum \mathbf{F}$$

which states that the rate of change of momentum of the particle is equal to the sum of the various forces acting on the particle. In this instance, the forces are those associated with friction and those related to gravity. Specifically, since m is a constant, we may write

$$\left(\frac{4}{3}\pi R^3 \rho_{\text{sph}}\right) \frac{d}{dt} \mathbf{v} = -\left(\pi R^2\right) \left(\frac{1}{2} \rho_{\text{air}} (v_x^2 + v_y^2) f\right) \frac{\mathbf{v}}{|\mathbf{v}|} + \left(\frac{4}{3}\pi R^3 \rho_{\text{sph}} - \frac{4}{3}\pi R^3 \rho_{\text{air}}\right) g \delta_y$$

Here the positive direction for the y axis has been taken to be in the direction of gravity (i.e., "downward"). The frictional force has been taken from Eqs. 6.1-5 and 5a, and the gravitational force from Eq. 6.1-6. Dividing \mathbf{v} by its absolute value $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$ gives the vector \mathbf{n} of Eq. 6.1-5a. When the above equation is divided by the mass of the sphere, we get

$$\frac{d}{dt} \mathbf{v} = -\frac{3}{8R} \frac{\rho_{\text{air}}}{\rho_{\text{sph}}} \sqrt{v_x^2 + v_y^2} f \mathbf{v} + \left(1 - \frac{\rho_{\text{air}}}{\rho_{\text{sph}}}\right) g \delta_y$$

Next we take the x and y components of this equation to get:

$$\frac{dv_x}{dt} = -\frac{3}{8R} \frac{\rho_{\text{air}}}{\rho_{\text{sph}}} v_x \sqrt{v_x^2 + v_y^2} f$$

$$\frac{dv_y}{dt} = -\frac{3}{8R} \frac{\rho_{\text{air}}}{\rho_{\text{sph}}} v_y \sqrt{v_x^2 + v_y^2} f + \left(1 - \frac{\rho_{\text{air}}}{\rho_{\text{sph}}}\right) g$$

in which it is understood that f is a function of the Reynolds number based on the instantaneous velocity: $\text{Re} = 2R|\mathbf{v}|\rho_{\text{air}}/\mu_{\text{air}}$.

b. For the sphere that is dropped vertically, the y component of the equation of motion becomes

$$\frac{dv_y}{dt} = -\frac{3}{8R} \frac{\rho_{\text{air}}}{\rho_{\text{sph}}} v_y^2 f + \left(1 - \frac{\rho_{\text{air}}}{\rho_{\text{sph}}}\right) g$$

Thus the particle that is fired horizontally has a frictional force that is $\sqrt{v_x^2 + v_y^2}/v_y$ times greater than that of the particle that moves vertically. Therefore the horizontally fired particle will be slowed down in its descent and the vertically moving particle will land first.

c. For the particle moving in the Stokes' law region, the

$$f = \frac{24}{\text{Re}} = \frac{24}{2R\sqrt{v_x^2 + v_y^2}\rho_{\text{air}}/\mu_{\text{air}}}$$

Then, the x and y components of the equation of motion given in (a) become:

$$\frac{dv_x}{dt} = -\frac{9}{2R^2} \frac{\mu_{\text{air}}}{\rho_{\text{sph}}} v_x$$

$$\frac{dv_y}{dt} = -\frac{9}{2R^2} \frac{\mu_{\text{air}}}{\rho_{\text{sph}}} v_y + \left(1 - \frac{\rho_{\text{air}}}{\rho_{\text{sph}}}\right) g$$

Thus, the equations of motion are "uncoupled." Therefore the motion in the y direction is completely independent of the motion in the x direction, and both particles will arrive at their destination simultaneously.

6D.1 Friction factor for a bubble in a clean liquid

The rate of energy dissipation in any region is, for a Newtonian fluid with negligible dilatational viscosity

$$\begin{aligned} E_v &= \iiint (-\boldsymbol{\tau} : \nabla \mathbf{v}) dV = \mu \iiint (\nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger) : \nabla \mathbf{v} dV \\ &= \frac{1}{2} \iiint (\nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger) : (\nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger) dV \end{aligned}$$

For potential flow, $[\nabla \times \mathbf{v}] = 0$ or $\nabla \mathbf{v} = (\nabla \mathbf{v})^\dagger$, and hence

$$E_v = 2\mu \iiint (\nabla \mathbf{v} : (\nabla \mathbf{v})^\dagger) dV = 2\mu \iiint (\nabla \cdot [\mathbf{v} \cdot (\nabla \mathbf{v})^\dagger]) dV - 2\mu \iiint (\mathbf{v} \cdot \nabla^2 \mathbf{v}) dV$$

But for irrotational flow of an incompressible fluid

$$(\nabla^2 \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - [\nabla \times [\nabla \times \mathbf{v}]] = 0$$

Then using the Gauss divergence theorem on the remaining term

$$\begin{aligned} E_v &= 2\mu \iint (\mathbf{n} \cdot [\mathbf{v} \cdot (\nabla \mathbf{v})^\dagger]) dS = \mu \iint (\mathbf{n} \cdot \nabla v^2) dS \\ &= -\mu \iint_{\text{surface at } R} \frac{\partial v^2}{\partial r} R^2 \sin \theta d\theta d\phi \end{aligned}$$

The surface integral includes the spherical surface at $r = R$ at which $\mathbf{n} = -\boldsymbol{\delta}_r$, and the surface at $r = \infty$ at which $\mathbf{n} = \boldsymbol{\delta}_r$.

For the potential flow around the sphere, we get from Eqs. 4B.5-2 and 3

$$v^2 = v_\infty^2 \left(\left[1 - \left(\frac{R}{r} \right)^3 \right]^2 \cos^2 \theta + \left[1 - \frac{1}{2} \left(\frac{R}{r} \right)^3 \right]^2 \sin^2 \theta \right)$$

From this we find

$$\left. \frac{\partial v^2}{\partial r} \right|_{r=R} = -\frac{9}{2} \frac{v_\infty^2}{R} \sin^2 \theta$$

Then, when this is substituted into the expression for E_v , we get

$$E_v = -2\pi\mu \cdot \left(-\frac{9}{2} \frac{v_\infty^2}{R} \right) R^2 \int_0^\pi \sin^3 \theta d\theta = 12\pi\mu R v_\infty^2 = F_k v_\infty$$

Hence, from $F_k = (\pi R^2) \left(\frac{1}{2} \rho v_\infty^2 \right) f$ we conclude that

$$f = \frac{12\pi\mu R v_\infty}{(\pi R^2) \left(\frac{1}{2} \rho v_\infty^2 \right)} = \frac{48}{2R v_\infty \rho / \mu} = \frac{48}{\text{Re}}$$

7A.1 Pressure rise in a sudden enlargement.

According to Eq. 7.6-4

$$p_2 - p_1 = \rho v_2^2 \left(\frac{1}{\beta} - 1 \right) \text{ in which } \beta = \frac{S_1}{S_2}$$

We compute β and v_2 thus:

$$\beta = \frac{S_1}{S_2} = \left(\frac{D_1}{D_2} \right)^2 = \left(\frac{5}{9} \right)^2 = 0.309$$

$$v_2 = \frac{Q}{S_2} = \frac{(450/60 \text{ gal/s})(0.13368 \text{ ft}^3/\text{gal})}{((\pi/4)(25/144) \text{ ft}^2)} = 7.35 \text{ ft/s}$$

Hence,

$$\begin{aligned} p_2 - p_1 &= \rho v_2^2 \left(\frac{1}{\beta} - 1 \right) \\ &= (63 \text{ lb}_m/\text{ft}^2)(7.35 \text{ ft/s})^2 \left(\frac{1}{0.309} - 1 \right) \\ &= 7610 \text{ poundals/ft}^2 = 1.64 \text{ psi} \end{aligned}$$

7A.2 Pumping a hydrochloric acid solution.

Equation 7.5-10 reduces to

$$\widehat{W}_m = \frac{p_2 - p_1}{\rho} + \frac{1}{2}v^2 \frac{L}{R_h} f \text{ (pipe friction)} + \frac{1}{2}v_2^2 \text{ (exit loss)}$$

for the liquid between free surface "1" and free surface "2", when the liquid is regarded as incompressible and the negligibility of kinetic and potential energy differences between those locations is noted. The major right-hand term is

$$\frac{p_2 - p_1}{\rho} = \frac{(4 - 1 \text{ atm})(6.8087 \times 10^4 \text{ lb}_m/\text{ft} \cdot \text{s}^2/\text{atm})}{62.4 \text{ lb}_m/\text{ft}^3} = 3273 \text{ ft}^2/\text{s}^2,$$

the pipe friction term (with $f = 0.0048_4$ from Eq. 6.2-12) is

$$\frac{1}{2}v^2 \frac{L}{D/4} f = \frac{1}{2}(2.30 \text{ ft/s})^2 \frac{300 \text{ ft}}{0.5/12 \text{ ft}} (0.0048_4) = 92 \text{ ft}^2/\text{s}^2,$$

and the exit loss is

$$\frac{1}{2}(2.30 \text{ ft/s})^2 = 2.6 \text{ ft}^2/\text{s}^2.$$

Thus, the required work input per unit mass of fluid pumped is

$$\widehat{W}_m = 3273 + 92 + 2.6 = 3368 \text{ ft}^2/\text{s}^2$$

Multiplication by the mass flow rate of

$$w = (\pi R^2) \rho \langle v \rangle = (\pi) \left(\frac{2^2}{144} \text{ ft}^2 \right) (62.4 \text{ lb}_m/\text{ft}^3) (2.30 \text{ ft/s}) = 12.5 \text{ lb}_m/\text{s}$$

gives the power required from the pump:

$$\begin{aligned} W &= \widehat{W}_m w = (3368 \text{ ft}^2/\text{s}^2)(12.5 \text{ lb}_m/\text{s}) \\ &= (4.22 \times 10^4 \text{ ft poundals/s}) \times (3600 \text{ s/hr})(1.5698 \times 10^{-8} \text{ hp-hr/ft poundal}) \\ &= 2.4 \text{ hp} = 1.8 \text{ kw} \end{aligned}$$

7A.3 Compressible gas flow in a cylindrical pipe.

For steady isothermal, horizontal pipe flow of an ideal gas at $T = 25^\circ\text{C} = 536.7 \text{ R}$ with flat velocity profiles and $\dot{W}_m = 0$, Eq. 7.4-7 gives

$$\begin{aligned}\hat{E}_v &= - \int_1^2 \frac{1}{\rho} dp - \left(\frac{v_2^2}{2} - \frac{v_1^2}{2} \right) \\ &= \frac{RT}{M} \ln \frac{p_1}{p_2} + \frac{1}{2} \left(\frac{w}{S} \right)^2 \left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} \right)\end{aligned}$$

Calculation of the terms in ft^2/s^2 gives

$$\frac{RT}{M} \ln \frac{p_1}{p_2} = \frac{(4.9686 \times 10^4 \text{ lb}_m \text{ft}^2/\text{s}^2 \cdot \text{lb-mol} \cdot \text{R})(536.7 \text{ R})}{(28.01 \text{ lb}_m/\text{lb-mol})} \ln 2 = 6.60 \times 10^5 \text{ ft}^2/\text{s}^2$$

and with $\rho = pM/RT$ in corresponding units,

$$\begin{aligned}\rho_1 &= (2 \times 2116.2 \text{ lb}_f/\text{ft}^2)(28.01 \text{ lb}_m/\text{lb-mol}) / (1544.3 \text{ ft} \cdot \text{lb}_f/\text{lb-mol} \cdot \text{R})(536.7 \text{ R}) \\ &= 0.1430 \text{ lb}_m/\text{ft}^3 = 2\rho_2\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{2} \left(\frac{w}{S} \right)^2 \left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} \right) &= \frac{1}{2} \left(\frac{(0.28 \text{ lb}_m/\text{s})}{(\pi/64 \text{ ft}^2)} \right)^2 \left(\frac{1}{(0.1430 \text{ lb}_m/\text{ft}^3)^2} - \frac{1}{(0.0715 \text{ lb}_m/\text{ft}^3)^2} \right) \\ &= -2.39 \times 10^3 \text{ ft}^2/\text{s}^2\end{aligned}$$

Combining these results, we get

$$\hat{E}_v = 6.60 \times 10^5 - 2.39 \times 10^3 = 6.58 \times 10^5 \text{ ft}^2/\text{s}^2$$

With the aid of Table F.3-3 we get the corresponding values in other units:

$$\hat{E}_v = 26.3 \text{ Btu}/\text{lb}_m = 6.11 \times 10^4 \text{ J}/\text{kg}$$

7A.4 Incompressible flow in an annulus.

(i) The mean hydraulic radius is (see Eq. 6.2-16):

$$R_h = \frac{S}{Z} = \frac{\pi(R_o^2 - R_i^2)}{2\pi(R_o + R_i)} = \frac{1}{2}(R_o - R_i)$$

The mass flow rate is

$$w = (241/60 \text{ U.S. gal/s})(231/1728 \text{ ft}^3/\text{gal})(62.4 \text{ lb}_m/\text{ft}^3) = 33.5 \text{ lb}_m/\text{s}$$

and the average flow velocity is

$$\begin{aligned} \langle v \rangle &= \frac{w}{\rho S} = \frac{w/\rho}{\pi(R_o^2 - R_i^2)} \\ &= \frac{(33.5 \text{ lb}_m/\text{s})/(62.4 \text{ lb}_m/\text{ft}^3)}{\pi((7/12)^2 - (3/12)^2 \text{ ft}^2)} = 0.615 \text{ ft/s} \end{aligned}$$

(ii) The Reynolds number is (see Eq. 6.2-18):

$$\begin{aligned} \text{Re} &= \frac{4R_h \langle v \rangle \rho}{\mu} = \frac{2(R_o - R_i)w}{\pi(R_o^2 - R_i^2)\mu} = \frac{2w}{\pi(R_o + R_i)\mu} \\ &= \frac{2(33.5 \text{ lb}_m/\text{s})}{\pi((7 + 3)/12 \text{ ft})(1.14 \times 0.000672 \text{ lb}_m/\text{ft}\cdot\text{s})} \\ &= 3.34 \times 10^4 \end{aligned}$$

Hence, the flow is turbulent and $f = 0.0059$ according to either Eq. 6.2-15 or Fig. 6.2-2.

(iii) From Eq. 7.5-10 we calculate the work required W in ft-pounds per pound [=] ft^2/s^2 :

$$\begin{aligned} \hat{W} &= g(h_2 - h_1) + \frac{1}{2}\langle v \rangle^2 \frac{L}{R_h} f \\ &= g(h_2 - h_1) + \langle v \rangle^2 \frac{L}{(R_o - R_i)} f \\ &= (32.2 \text{ ft/s}^2)(5 \text{ ft}) + (0.615 \text{ ft/s})^2 \left(\frac{20.3}{(7 - 3)/12} \right) (0.0059) \\ &= (161 + 0.175) = 161.1 \text{ ft}^2/\text{s}^2 \end{aligned}$$

The power output required from the pump is then

$$\begin{aligned} w\hat{W} &= (33.5 \text{ lb}_m/\text{s})(161.1 \text{ ft}^2/\text{s}^2) = 5.4 \times 10^3 \text{ ft}\cdot\text{pounds}/\text{s} \\ &= 1.9 \times 10^7 \text{ ft}\cdot\text{pounds}/\text{hr} = 0.31 \text{ hp} = 0.23 \text{ kw} \end{aligned}$$

7A.5 Force on a U-bend.

According to Eq. 7.2-3, if the x -axis is taken in the direction of the downstream unit vector \mathbf{u}_1 at S_1 , which is the negative of the downstream unit vector \mathbf{u}_2 at S_2 and perpendicular to the gravity vector \mathbf{g} , then the force exerted by the water on the U-bend has the x -component

$$(\boldsymbol{\delta}_x \bullet \mathbf{F}_{f \rightarrow s}) = (v_1 w + p_1 S_1)(\boldsymbol{\delta}_x \bullet \mathbf{u}_1) - (v_2 w + p_2 S_2)(\boldsymbol{\delta}_x \bullet \mathbf{u}_2) + m_{\text{tot}}(\boldsymbol{\delta}_x \bullet \mathbf{g})$$

when the velocity profiles at Planes 1 and 2 are approximated as flat. With $(\boldsymbol{\delta}_x \bullet \mathbf{u}_1) = 1$, $(\boldsymbol{\delta}_x \bullet \mathbf{u}_2) = -1$, $(\boldsymbol{\delta}_x \bullet \mathbf{g}) = 0$, $S_1 = S_2$, and $v_1 = v_2 = w/\rho = Q/S$, this gives

$$\begin{aligned} F_{x, f \rightarrow s} &= (v_1 + v_2)w + (p_1 + p_2)S \\ &= (2Q/S)(Q\rho) + (p_1 + p_2)S \\ &= 2Q^2\rho/S + (p_1 + p_2)S \end{aligned}$$

with $Q = 3 \text{ ft}^3/\text{s}$. Thus, the horizontal force of the water on the U-bend is

$$\begin{aligned} F_{x, f \rightarrow s} &= 2(3 \text{ ft}^3/\text{s})^2(62.4 \text{ lb}_m/\text{ft}^3)/(4\pi/144 \text{ ft}^2) \\ &\quad + (21 + 19 \text{ lb}_f/\text{in}^2)(4\pi \text{ in}^2)(32.174 \text{ lb}_m\text{ft}/\text{s}^2 \cdot \text{lb}_f) \\ &= (12871 + 16172 \text{ lb}_m\text{ft}/\text{s}^2) = 29043 \text{ lb}_m\text{ft}/\text{s}^2 = 903 \text{ lb}_f \end{aligned}$$

7A.6 Flow-rate calculation.

As our system for the mechanical energy balance, we consider the water between plane 1 (the upper liquid surface in the constant-head tank) and plane 2 (just inside the outlet of the exit piping). We neglect the velocity at plane 1, set $p = p_{\text{atm}}$ at planes 1 and 2, and treat the water as incompressible. Then Equation 7.5-10 takes the form

$$\frac{1}{2}v_2^2 + g(z_2 - z_1) = -\frac{1}{2}v^2 \frac{L}{R_h} f - \frac{1}{2}v^2 \sum_i e_{vi}$$

in which the kinetic energy and friction terms all are calculated with the same average velocity v .

Collecting the coefficients of v^2 , we get

$$v^2 \left[1 + \frac{L}{R_h} f + 0.45 + 0.4 + 0.4 \right] = 2g(z_1 - z_2)$$

in which the coefficients 0.45, 0.4, and 0.4 are the e_{vi} values for the sudden contraction at the pipe inlet and for two smooth 90° elbows. There is no enlargement loss in the system considered, since plane 2 is just inside the exit of the piping. Setting $R_h = D/4$ and solving for v , we get

$$v = \sqrt{\frac{2g(z_1 - z_2)}{2.25 + 4fL/D}}$$

This result is explicit except for the friction factor f , which depends on Re and thus on v .

Multiplying the last equation by v/ν and evaluating all the dimensional constants in cgs, we get the working formula

$$\begin{aligned} \text{Re} &= \frac{Dv}{\nu} = \frac{D}{\nu} \sqrt{\frac{2g(z_1 - z_2)}{2.25 + 4L/Df}} \\ &= \frac{12.7 \text{ cm}}{0.010037 \text{ cm}^2/\text{s}} \sqrt{\frac{2(980.7 \text{ cm/s}^2)(38 \times 30.48 \text{ cm})}{2.25 + 355.2f}} \\ &= \frac{1.907 \times 10^6}{\sqrt{2.25 + 355.2f}} \end{aligned} \quad (\text{A})$$

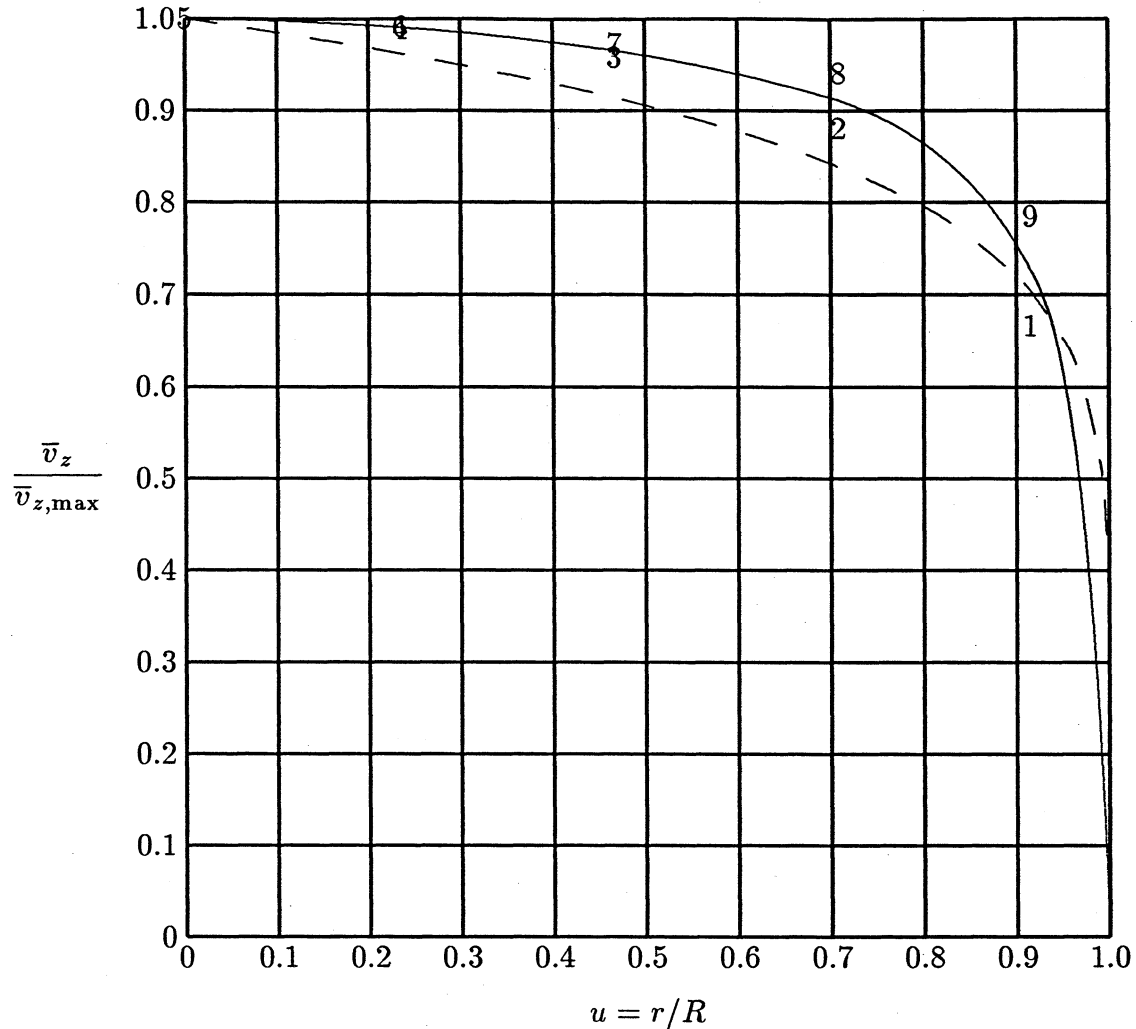
The last equation is solvable rapidly by iteration. A first trial value 1×10^5 for Re gives $f = 0.0045$ from Fig. 6.2.2, and a new value $\text{Re} = 9.72 \times 10^5$ from Eq. (A). At this new trial Re value, we get $f = 0.00285$ from Fig. 6.2-2, and Eq. (A) then gives $\text{Re} = 1.056 \times 10^6$. Returning to Fig. 6.2-2 with this new trial Re value, we get $f = 0.0028$, and Eq. (A) then gives the value $\text{Re} = 1.059 \times 10^6$, which we accept as final.

The mass flow rate can then be calculated as $w = \text{Re}\pi D\mu/4$, or the volume flow rate as

$$Q = \frac{w}{\rho} = \frac{\text{Re}\pi D\mu}{4}$$
$$= \frac{(1.056 \times 10^6)(\pi)(12.7 \text{ cm})(0.010037 \text{ cm}^2/\text{s})}{4} = 1.1 \times 10^5 \text{ cm}^3/\text{s} = 0.11 \text{ m}^3/\text{s}$$

Calibration is necessary to get accurate measurements of flow rates in such an apparatus, because of the uncertainties in the geometric details and thus in the energy loss factors (see Table 7.5-1).

7A.7 Evaluation of various velocity averages from Pitot tube data.



This figure shows velocity data from the C. E. thesis of B. Bird (University of Wisconsin, 1915). Each measurement is labelled with its position number from the table in Problem 7A.7. The flow is evidently turbulent, but not fully developed; if it were, the measured velocities at positions 1, 2, and 3 would show better agreement with those at positions 7, 8, and 9 for the same r values. The solid curve in the figure is a curve-fit of the data, and the dashed curve represents the 1/7th-power model given in Eq. 5.1-4, which is based on extensive experiments in fully developed flow.

Application of Simpson's rule to 11 uniformly spaced points on the solid curve gives the following velocity averages. A finer grid would give more accurate approximations to the integrals.

$$\begin{aligned}
 \frac{\langle v \rangle}{v_{\max}} &= \int_0^1 \frac{v}{v_{\max}} 2u du \\
 &= [(1 \times 1.0 \times 0) + (4 \times 0.995 \times 0.1) + (2 \times 0.986 \times 0.2) \\
 &\quad + (4 \times 0.98 \times 0.3) + (2 \times 0.975 \times 0.4) + (4 \times 0.96 \times 0.5) \\
 &\quad + (2 \times 0.94 \times 0.6) + (4 \times 0.92 \times 0.7) + (2 \times 0.87 \times 0.8) \\
 &\quad + (4 \times 0.755 \times 0.9) + (1 \times 0 \times 1.0)] \times 2/30 \\
 &= 0.832
 \end{aligned}$$

$$\begin{aligned}
 \frac{\langle v^2 \rangle}{v_{\max}^2} &= \int_0^1 \frac{v^2}{v_{\max}^2} 2u du \\
 &= [(1 \times 1.0^2 \times 0) + (4 \times 0.995^2 \times 0.1) + (2 \times 0.986^2 \times 0.2) \\
 &\quad + (4 \times 0.98^2 \times 0.3) + (2 \times 0.975^2 \times 0.4) + (4 \times 0.96^2 \times 0.5) \\
 &\quad + (2 \times 0.94^2 \times 0.6) + (4 \times 0.92^2 \times 0.7) + (2 \times 0.87^2 \times 0.8) \\
 &\quad + (4 \times 0.755^2 \times 0.9) + (1 \times 0^2 \times 1.0)] \times 2/30 \\
 &= 0.749
 \end{aligned}$$

$$\begin{aligned}
 \frac{\langle v^3 \rangle}{v_{\max}^3} &= \int_0^1 \frac{v^3}{v_{\max}^3} 2u du \\
 &= [(1 \times 1.0^3 \times 0) + (4 \times 0.995^3 \times 0.1) + (2 \times 0.986^3 \times 0.2) \\
 &\quad + (4 \times 0.98^3 \times 0.3) + (2 \times 0.975^3 \times 0.4) + (4 \times 0.96^3 \times 0.5) \\
 &\quad + (2 \times 0.94^3 \times 0.6) + (4 \times 0.92^3 \times 0.7) + (2 \times 0.87^3 \times 0.8) \\
 &\quad + (4 \times 0.755^3 \times 0.9) + (1 \times 0^3 \times 1.0)] \times 2/30 \\
 &= 0.680
 \end{aligned}$$

These integrals give the ratios $\langle v^2 \rangle / \langle v \rangle^2 = 1.08$ and $\langle v^3 \rangle / \langle v \rangle^3 = 1.18$. As one might expect from inspection of the plotted curves, these ratios differ significantly from the values 1.02 and 1.06 calculated for a 1/7th-power velocity profile.

7B.1 Velocity averages from the $\frac{1}{7}$ -power law

a. Average of the velocity

$$\begin{aligned}\frac{\langle \bar{v}_z \rangle}{v_{\max}} &= \frac{\int_0^{2\pi} \int_0^R [1 - (r/R)]^{1/7} r dr d\theta}{\int_0^{2\pi} \int_0^R r dr d\theta} = 2 \int_0^1 (1 - \xi)^{1/7} \xi d\xi \\ &= 2 \int_0^1 x(1 - x^7)(7x^6) dx = 14 \left(\frac{1}{8} - \frac{1}{15} \right) = \frac{49}{60}\end{aligned}$$

b. Average of the square of the velocity

$$\begin{aligned}\frac{\langle \bar{v}_z^2 \rangle}{v_{\max}^2} &= \frac{\int_0^{2\pi} \int_0^R [1 - (r/R)]^{2/7} r dr d\theta}{\int_0^{2\pi} \int_0^R r dr d\theta} = 2 \int_0^1 (1 - \xi)^{2/7} \xi d\xi \\ &= 2 \int_0^1 x(1 - x^{7/2}) \left(\frac{7}{2} x^{5/2} \right) dx = 7 \left(\frac{2}{9} - \frac{1}{8} \right) = \frac{49}{72} \\ \text{so that } \langle \bar{v}_z^2 \rangle / \langle \bar{v}_z \rangle^2 &= \left(\frac{49}{72} \right) \left(\frac{60}{49} \right)^2 = \frac{50}{49}\end{aligned}$$

c. Average of the cube of the velocity

$$\begin{aligned}\frac{\langle \bar{v}_z^3 \rangle}{v_{\max}^3} &= \frac{\int_0^{2\pi} \int_0^R [1 - (r/R)]^{3/7} r dr d\theta}{\int_0^{2\pi} \int_0^R r dr d\theta} = 2 \int_0^1 (1 - \xi)^{3/7} \xi d\xi \\ &= 2 \int_0^1 x(1 - x^{7/3}) \left(\frac{7}{3} x^{4/3} \right) dx = \frac{14}{3} \left(\frac{3}{10} - \frac{3}{17} \right) = \frac{49}{85} \\ \text{so that } \langle \bar{v}_z^3 \rangle / \langle \bar{v}_z \rangle^3 &= \left(\frac{49}{85} \right) \left(\frac{60}{49} \right)^3 = \frac{43200}{40817}\end{aligned}$$

7B.2 Relation between force and viscous loss for flow in conduits of variable cross section

For the variable cross-section, the macroscopic balances in Eqs. 7.5-5 and 6 have to be restated (when the gravitational forces can be neglected) thus:

$$\text{(z-momentum)} \quad F_{f \rightarrow s} = (v_1 - v_2)w + (p_1 S_1 - p_2 S_2)$$

$$\text{(mechanical energy)} \quad \hat{E}_v = \frac{1}{2}(v_1^2 - v_2^2) + \frac{1}{\rho}(p_1 - p_2)$$

To prove that Eq. 7B.2-1 is correct, we multiply the second equation above by ρS_m (defined in the problem statement) to get

$$\begin{aligned} \rho S_m \hat{E}_v &= \frac{1}{2} \rho S_m (v_1^2 - v_2^2) + S_m (p_1 - p_2) \\ &= \frac{1}{2} \rho S_m (v_1 - v_2)(v_1 + v_2) + S_m (p_1 - p_2) \end{aligned}$$

If Eq. 7B.2-1 is to be correct, then in the term containing the velocities, we must have $w = \frac{1}{2} \rho S_m (v_1 + v_2)$. That this is indeed true can be seen as follows:

$$\frac{1}{2} \rho S_m (v_1 + v_2) = \rho \left(\frac{w}{\rho S_1} + \frac{w}{\rho S_2} \right) \left(\frac{S_1 S_2}{S_1 + S_2} \right) = \rho \left(\frac{w}{\rho} \right) \left(\frac{1}{S_1} + \frac{1}{S_2} \right) \left(\frac{S_1 S_2}{S_1 + S_2} \right) = w$$

Next, we have to verify that $(p_1 S_1 - p_2 S_2) = S_m (p_1 - p_2) + p_m (S_1 - S_2)$. Substituting the definitions of S_m and p_m gives for the right side:

$$\begin{aligned} &\left(\frac{2S_1 S_2}{S_1 + S_2} \right) (p_1 - p_2) + \left(\frac{p_1 S_1 + p_2 S_2}{S_1 + S_2} \right) (S_1 - S_2) \\ &= \frac{2S_1 S_2 p_1 - 2S_1 S_2 p_2 + S_1^2 p_1 - S_2^2 p_2 - S_1 S_2 p_1 + S_1 S_2 p_2}{S_1 + S_2} \\ &= \frac{p_1 S_1 (S_1 + S_2) - p_2 S_2 (S_1 + S_2)}{S_1 + S_2} = p_1 S_1 - p_2 S_2 \end{aligned}$$

(Ref: R. B. Bird, *CEP Symposium Ser. #58*, Vol. 61 (1965), pp. 14-15.)

7B.3 Flow through a sudden enlargement

The pressure rise is given by Eq. 7.6-5, and, in Eq. 7.6-6, in terms of the downstream velocity. We need a similar expression written in terms of the upstream velocity:

$$p_2 - p_1 = \rho(\beta v_1)[v_1 - (\beta v_1)] = \rho v_1^2 \beta(1 - \beta)$$

Then, to get the extremum, we set the first derivative equal to zero:

$$\frac{d(p_2 - p_1)}{d\beta} = \rho v_1^2 (1 - 2\beta) = 0$$

Hence, we get

$$\beta = \frac{S_1}{S_2} = \frac{D_1^2}{D_2^2} = \frac{1}{2} \quad \text{or} \quad \frac{D_2}{D_1} = \sqrt{2}$$

We still have to verify that this is a maximum for the velocity. To do that, we need the second derivative:

$$\frac{d^2(p_2 - p_1)}{d\beta^2} = -2\rho v_1^2 < 0$$

Since this is negative, we have found the location of the maximum. The maximum pressure difference is then

$$p_2 - p_1 = \rho v_1^2 \beta(1 - \beta) = \frac{1}{4} \rho v_1^2$$

7B.4 Flow between two tanks

The flow in a tube can be described by the steady-state macroscopic energy balance of Eq. 7.4-7, which for this problem simplifies to

$$\frac{p_B - p_A}{\rho} = -\hat{E}_v \quad \text{or} \quad p_A - p_B = \rho \hat{E}_v$$

Then using Eqs. 7.5-8, 6.2-16, and 6.2-12 we get

$$p_A - p_B = \frac{1}{2} \rho \langle v^2 \rangle \frac{L}{R_h} f = \frac{1}{2} \rho \langle v^2 \rangle \frac{4L}{D} f = \frac{1}{2} \rho \langle v^2 \rangle \frac{4L}{D} \cdot \frac{0.0791}{\text{Re}^{1/4}}$$

(Note that we could just as well have started with Eqs. 6.1-4 and 6.2-12.) The last equation can now be written for each of the two systems, and we omit the factors involving numerical constants, tube length, and the physical properties (since they are the same for the two systems):

$$\begin{aligned} (p_A - p_B)_I &\propto \left(\frac{w_I}{S_I} \right)^2 \frac{1}{D_I} \left(\frac{D_I w_I}{S_I} \right)^{-1/4} \propto \left(\frac{w_I}{S_I} \right)^2 \frac{1}{\sqrt{S_I}} \left(\frac{w_I}{\sqrt{S_I}} \right)^{-1/4} \\ (p_A - p_B)_{II} &\propto \left(\frac{w_{II}}{S_{II}} \right)^2 \frac{1}{D_{II}} \left(\frac{D_{II} w_{II}}{S_{II}} \right)^{-1/4} \propto \left(\frac{w_{II}}{S_{II}} \right)^2 \frac{1}{\sqrt{S_{II}}} \left(\frac{w_{II}}{\sqrt{S_{II}}} \right)^{-1/4} \\ &\propto \left(\frac{w_I}{S_I} \right)^2 \frac{1}{\sqrt{\frac{1}{2} S_I}} \left(\frac{\frac{1}{2} w_I}{\sqrt{\frac{1}{2} S_I}} \right)^{-1/4} \end{aligned}$$

In the last step, we have used $S_{II} = \frac{1}{2} S_I$ and $w_{II} = \frac{1}{2} w_I$. Hence

$$\frac{(p_A - p_B)_{II}}{(p_A - p_B)_I} = \frac{1}{\sqrt{\frac{1}{2}}} \left(\frac{\frac{1}{2}}{\sqrt{\frac{1}{2}}} \right)^{-1/4} = \sqrt{2} (\sqrt{2})^{1/4} = 2^{(1/2)+(1/8)} = 2^{5/8}$$

7B.5 Revised design of an air duct

a. The flow in a tube can be described by the steady-state macroscopic energy balance of Eq. 7.4-7, which for this problem simplifies to

$$\frac{p_2 - p_1}{\rho} = -\hat{E}_v \quad \text{or} \quad p_1 - p_2 = \rho \hat{E}_v$$

Then using Eqs. 7.5-8, 6.2-16, and 6.2-12 we get

$$p_1 - p_2 = \frac{1}{2} \rho \langle v^2 \rangle \frac{L}{R_h} f = \frac{1}{2} \rho \langle v^2 \rangle \frac{4L}{D} f = \frac{1}{2} \rho \langle v^2 \rangle \frac{4L}{D} \cdot \frac{0.0791}{\text{Re}^{1/4}}$$

(Note that we could also have started with Eqs. 6.1-4 and 6.2-12.)

b. Label the two systems--the original and the revised--I and II. Then the pressure drop in the two systems will be

$$(p_1 - p_2)_I = \frac{0.0791}{(4R_{h,I} \langle v \rangle_I \rho / \mu)^{1/4}} \frac{\frac{1}{2} \rho \langle v^2 \rangle_I L}{R_{h,I}}$$

$$(p_1 - p_2)_{II} = \frac{0.0791}{(4R_{h,II} \langle v \rangle_{II} \rho / \mu)^{1/4}} \frac{\frac{1}{2} \rho \langle v^2 \rangle_{II} L}{R_{h,II}}$$

Since the numerical constant, the length L , and the values of the physical constants are exactly the same, we now omit those quantities from consideration and concentrate on the quantities that are different. Then we have (with W and H being used for the width and height of the ducts):

$$(p_1 - p_2)_I \propto \frac{\langle v \rangle_I^{7/4}}{R_{h,I}^{5/4}} \propto \frac{(w/S_I)^{7/4}}{(S_I/Z_I)^{5/4}} \propto \frac{Z_I^{5/4}}{S_I^3} = \frac{[2(W_I + H_I)]^{5/4}}{[W_I H_I]^3}$$

$$(p_1 - p_2)_{II} \propto \frac{\langle v \rangle_{II}^{7/4}}{R_{h,II}^{5/4}} \propto \frac{(w/S_{II})^{7/4}}{(S_{II}/Z_{II})^{5/4}} \propto \frac{Z_{II}^{5/4}}{S_{II}^3} = \frac{[2(W_{II} + H_{II})]^{5/4}}{[W_{II} H_{II}]^3}$$

We have used the fact that w is the same in both systems. Next we make use of the fact that the pressure drop is the same in both systems:

$$\frac{[2(W_I + H_I)]^{5/4}}{[W_I H_I]^3} = \frac{[2(W_{II} + H_{II})]^{5/4}}{[W_{II} H_{II}]^3}$$

c. Next we put in numerical values

$$\frac{[2(4 + 4)]^{5/4}}{[4 \cdot 4]^3} = \frac{[2(W_{II} + 2)]^{5/4}}{[2W_{II}]^3}$$

Taking the fourth power of both sides gives

$$\frac{[16]^5}{[16]^{12}} = \frac{[2(W_{II} + 2)]^5}{[2W_{II}]^{12}}$$

Thus the equation for $2W_{II} = x$ becomes

$$\frac{1}{16^7} = \frac{(x + 4)^5}{x^{12}}$$

This equation can be solved by trial and error, and the solution is $x = 18.4$ ft or $W_{II} = 9.2$ ft.

7B.6 Multiple discharge into a common conduit

a. Application of the macroscopic mass balance gives:

$$w_1 = w_2 \quad \text{or} \quad \langle v_1 \rangle S_1 = \langle v_2 \rangle S_2 \quad (\text{for incompressible flow})$$

the relation between the average velocities is

$$\frac{\langle v_2 \rangle}{\langle v_1 \rangle} = \beta$$

We now define, for later use, a set of quantities $K_i^{(j)}$ as

$$K_i^{(j)} = \frac{\langle v_i^j \rangle}{\langle v_i \rangle^j}$$

where the index i indicates inlet (1) and (2), and j is the power to which the velocities are raised. For $j = 2$ and 3 , these quantities are (for the $1/7$ th power law of turbulent flow) $\frac{50}{49}$ and $\frac{43200}{40817}$ respectively (see Problem 7B.1)..

b. Application of the macroscopic momentum balance in the direction of flow gives (with no external forces):

$$\rho \langle v_1^2 \rangle S_1 - \rho \langle v_2^2 \rangle S_2 + p_1 S_1 - p_2 S_2 - F_{f \rightarrow s} = 0$$

The force of the fluid on the solid will consist of the viscous force

$$p_2 - p_1 = \rho \langle v_1 \rangle^2 (\beta K_1^{(2)} - \beta^2 K_2^{(2)})$$

c. Finally we apply the steady-state mechanical energy balance (Eq. 7.4-7) for incompressible fluids to get

$$\hat{E}_v = \frac{1}{2} \left(\frac{\langle v_1^3 \rangle}{\langle v_1 \rangle} - \frac{\langle v_2^3 \rangle}{\langle v_2 \rangle} \right) + \frac{p_1 - p_2}{\rho}$$

The pressure difference is evaluated from part (b), so that we get

$$\begin{aligned} \hat{E}_v &= \frac{1}{2} \langle v_1 \rangle^2 \left(\frac{\langle v_1^3 \rangle}{\langle v_1 \rangle^3} - \frac{\langle v_2^3 \rangle}{\langle v_2 \rangle^3} \frac{\langle v_2 \rangle^2}{\langle v_1 \rangle^2} \right) - \langle v_1 \rangle^2 (\beta K_1^{(2)} - \beta^2 K_2^{(2)}) \\ &= \frac{1}{2} \langle v_1 \rangle^2 K_1^{(3)} \left(1 - \beta^2 \frac{K_2^{(3)}}{K_1^{(3)}} \right) - \langle v_1 \rangle^2 (\beta K_1^{(2)} - \beta^2 K_2^{(2)}) \\ &= \frac{1}{2} \langle v_1 \rangle^2 K_1^{(3)} \left(1 - 2\beta \frac{K_1^{(2)}}{K_1^{(3)}} + \beta^2 \left(2 \frac{K_2^{(2)}}{K_1^{(3)}} - \frac{K_2^{(3)}}{K_1^{(3)}} \right) \right) \end{aligned}$$

For *turbulent flow*, the ratios of the $K_i^{(j)}$ in these expressions are of the order of unity.

7B.7 Inventory variations in a gas reservoir

a. The maximum, minimum, and average values of w_2 are

$$(w_2)_{\max} = A + B \qquad (w_2)_{\min} = A - B$$

$$(w_2)_{\text{avg}} = \frac{\int_{\omega t=0}^{2\pi} (A + B \cos \omega t) d(\omega t)}{\int_{\omega t=0}^{2\pi} d(\omega t)} = A$$

b. A mass balance over a 24-hour period is obtained by integrating the unsteady mass balance over the 24-hour period:

$$\int_0^{24} \left(\frac{dm_{\text{tot}}}{dt} \right) dt = \int_0^{24} w_1 dt - \int_0^{24} w_2 dt$$

If the amount of gas in the reservoir is to have the same average value over a 24-hour period, then

$$0 = 24w_1 - 24A$$

from which $w_1 = A$.

c. The total mass of gas in the reservoir as a function of time can be found by integrating the unsteady macroscopic mass balance:

$$\int_0^t \left(\frac{dm_{\text{tot}}}{d\bar{t}} \right) d\bar{t} = \int_0^t w_1 d\bar{t} - \int_0^t w_2 d\bar{t}$$

to get

$$m_{\text{tot}}(t) = m_{\text{tot}}^0 + \int_0^t (A - A - B \cos \omega \bar{t}) d\bar{t} = m_{\text{tot}}^0 - \frac{B}{\omega} \sin \omega t$$

d. In this problem we are assuming that the density of the fluid does not appreciably change with time, and so we assume that it is constant. However, the volume of the gas, $V(t)$, will be a function of time and $m_{\text{tot}}(t) = \rho V(t)$, so that

$$\rho V(t) = m_{\text{tot}}^0 - \frac{B}{\omega} \sin \omega t$$

The criterion for uninterrupted operation of the system is that $V(t)$ must never go below zero. In this limiting situation, the quantity $m_{\text{tot}}(t) = \rho V(t)$ must then oscillate between the minimum value, $m_{\text{tot}}^0 - (B/\omega)$, and the maximum value, $m_{\text{tot}}^0 + (B/\omega)$. Therefore, the minimum total mass in the system that can accommodate this kind of oscillation is that

$$\rho V_{\text{min}} = [m_{\text{tot}}^0 + (B/\omega)] - [m_{\text{tot}}^0 - (B/\omega)]$$

or

$$V_{\text{min}} = \frac{2B}{\rho\omega} = \frac{(2)(2000)}{(0.044)(2\pi/24)} = 3.48 \times 10^5 \text{ ft}^3$$

3. Add a three-day supply to the amount found above

$$\begin{aligned} V_{\text{min}} &= \frac{2B}{\rho\omega} + (3 \times 24A) = (3.48 \times 10^5) + \frac{(72)(5000)}{(0.044)} \\ &= (3.48 \times 10^5) + (8.18 \times 10^6) = 8.53 \times 10^6 \text{ ft}^3 \end{aligned}$$

7B.8 Change in liquid height with time

a. We want to calculate the volume between the liquid level at h and the part of the sphere below the liquid level. The sphere is visualized as being generated by a circle in the xz plane, with its bottom at the origin and its center at $x = 0, z = R$. Such a circle has the equation

$$x^2 + (z - R)^2 = R^2$$

or

$$x^2 = 2Rz - z^2$$

Next we visualize the liquid volume as being made up of thin circular disks of thickness dz , each disk with a volume

$$dV = \pi x^2 dz = \pi(2Rz - z^2) dz$$

Then the total volume of the liquid is

$$\begin{aligned} V &= \pi \int_0^h (2Rz - z^2) dz \\ &= \pi \left(Rz^2 - \frac{1}{3} z^3 \right) \Big|_0^h \\ &= \pi \left(Rh^2 - \frac{1}{3} h^3 \right) \end{aligned}$$

or

$$V = \pi R h^2 \left(1 - \frac{1}{3} \frac{h}{R} \right)$$

This may be checked by verifying that when the tank is full, $h = 2R$, and the liquid volume is $V = \frac{4}{3} \pi R^3$; that when the tank is half full, $h = R$, and the liquid volume is $V = \frac{2}{3} \pi R^3$; and that when the tank is empty, $h = 0$, the liquid volume is $V = 0$

(Note: This method of obtaining the liquid volume was suggested by Professor L. E. Wedgewood, University of Illinois at Chicago)

b. To get the liquid height as a function of the time, we start with the differential equation in Eq. 7.1-7 which may be rearranged to give:

$$\left(H - 2(L + R) + \frac{L(2R + L)}{H} \right) \frac{dH}{dt} = A$$

This separable, first-order equation can be integrated to give

$$\frac{1}{2}H^2 - 2(L + R)H + L(2R + L)\ln H = At + C$$

At $t = 0$, $h = 2R$, and $H = 2R + L$. Therefore

$$\frac{1}{2}(2R + L)^2 - 2(L + R)(2R + L) + L(2R + L)\ln(2R + L) = C$$

Subtraction and elimination of the integration constant then gives

$$\frac{1}{2}\left[H^2 - (2R + L)^2 \right] - 2(L + R)[H - (2R + L)] + L(2R + L)\ln \frac{H}{2R + L} = At$$

or

$$\frac{1}{2}\left[(h + L)^2 - (2R + L)^2 \right] - 2(L + R)(h - 2R) + L(2R + L)\ln \frac{h + L}{2R + L} = At$$

When $h = 2R$, this equation gives $t = 0$, and when $h = 0$, it gives $t = t_{\text{efflux}}$ exactly as in Eq. 7.1-8. We now introduce dimensionless variables: $\eta = h/2R$ (dimensionless liquid height) and $\lambda = L/2R$ (dimensionless tube length). Then the above equation becomes:

$$\frac{1}{2}\left[(\eta + \lambda)^2 - (1 + \lambda)^2 \right] - (2\lambda + 1)(\eta - 1) + \lambda(1 + \lambda)\ln \frac{\eta + \lambda}{1 + \lambda} = \frac{At}{4R^2}$$

c. The parameter $\lambda = L/2R$ is fixed by the geometry of the system. Choose values of $\eta = h/2R$ from 0 to 1 and calculate t . from the above equation. These may be plotted to give the curve of the dimensionless liquid height versus the dimensionless time.

7B.9 Draining of a cylindrical tank with exit pipe

a. First we write the unsteady-state mass balance for the tank. The mass of fluid in the tank at any time is $\pi R^2 h \rho$, and hence the mass balance is (since there is no inflow stream)

$$\frac{d}{dt} \pi R^2 h \rho = -w_2$$

The quantity w_2 is the mass rate of flow out of the tank, and this is equal to the mass rate of flow in the tube. The latter is given by the Hagen-Poiseuille formula:

$$w_2 = \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)D^4 \rho}{128\mu L} = \frac{\pi(\rho g h + \rho g L)D^4 \rho}{128\mu L}$$

Here $\rho g h$ is the pressure p_0 exerted by the fluid above the tube entrance, and $\rho g L$ is the " $\rho g h$ -term" in the expression for \mathcal{P} in fn. 1 on p. 50 and discussed after Eq. 3.5-7 on p. 84. (It is unfortunate that we are dealing with two h 's here: the h in the expression for \mathcal{P} and the h which is the height of the fluid in the cylindrical tank; the h 's must not be confused with each other.)

The mass balance is then

$$\pi R^2 \rho \frac{dh}{dt} = -\frac{\pi \rho g (h+L) D^4 \rho}{128\mu L} \quad \text{or} \quad \frac{dh}{dt} = -\frac{g(h+L) D^4 \rho}{128\mu L R^2}$$

This first-order, separable equation can be integrated thus:

$$\int_H^0 \frac{dh}{h+L} = -\int_0^{t_{\text{efflux}}} \frac{\rho g D^4}{128\mu L R^2} dt \quad \text{or} \quad t_{\text{efflux}} = \frac{128\mu L R^2}{\rho g D^4} \ln\left(1 + \frac{H}{L}\right)$$

b. One has many mass-flow-rate vs. pressure-difference relations to choose from in turbulent flow. For purposes of illustration we use Eq. 5.1-6, which can be shown to be identical to the Blasius formula given in Eq. 6.2-12. Thus we replace the second equation in part (a) by

$$w = \frac{\pi}{2} \left(\frac{\rho R^{19/4}}{0.198 \mu^{1/4} L} \right)^{4/7} (\mathfrak{P}_0 - \mathfrak{P}_L)^{4/7} = \frac{\pi}{2} \left(\frac{\rho R^{19/4}}{0.198 \mu^{1/4} L} \right)^{4/7} [\rho g (h + L)]^{4/7}$$

Then the mass balance becomes

$$\frac{dh}{dt} = -\frac{1}{2} \left(\frac{\rho^{1/4} R^{5/4}}{0.198 \mu^{1/4} L} \right)^{4/7} (h + L)^{4/7} \equiv -B(h + L)^{4/7}$$

This equation can be integrated to give

$$\frac{7}{3}(h + L)^{3/7} = -Bt + C$$

Application of the initial condition gives

$$\frac{7}{3}(H + L)^{3/7} = C$$

Subtraction then gives the expression for the instantaneous liquid level

$$\frac{7}{3} \left[(H + L)^{3/7} - (h + L)^{3/7} \right] = Bt$$

The efflux time is then

$$\frac{7}{3} \left[(H + L)^{3/7} - L^{3/7} \right] = Bt_{\text{efflux}}$$

or

$$t_{\text{efflux}} = \frac{14}{3} \left(\frac{0.198 \mu^{1/4} L}{\rho^{1/4} R^{5/4}} \right)^{4/7} \left[(H + L)^{3/7} - L^{3/7} \right]$$

7B.10 Efflux time for draining a conical tank

a. From the unsteady mass balance we get, for the truncated cone system with no input stream

$$\frac{d}{dt} m_{\text{tot}} = -w_2 \quad \text{or} \quad \frac{d}{dt} \left(\frac{1}{3} \pi r^2 z \rho - \frac{1}{3} \pi r_2^2 z_2 \right) = -\rho v_2 (\pi r_2^2)$$

From Fig. 7B.10 we see that $r/r_2 = z/z_2$ so that the mass balance may be rewritten as

$$\frac{d}{dt} \left(\frac{1}{3} \pi z^3 \rho (r_2/z_2)^2 \right) = -\rho v_2 (\pi r_2^2) \quad \text{or} \quad \frac{d}{dt} \left(\frac{1}{3} z^3 \right) = -v_2 z_2^2$$

Therefore we get

$$z^2 \frac{dz}{dt} = -v_2 z_2^2 \quad \text{or} \quad v_2 = -\frac{z^2}{z_2^2} \frac{dz}{dt}$$

b. We simplify the unsteady mechanical energy balance by (i) omitting the kinetic energy contribution on the left side, because it was shown in Example 7.7-1 to be unimportant, (ii) neglecting the viscous dissipation term E_v because it is believed to be small, (iii) assuming incompressibility (which causes the E_c term to drop out, (iv) omitting the work term, since there are no moving parts, (v) omitting the pressure terms on the right side, since the pressure is atmospheric at both ends, and (vi) assuming no vortex motion of the fluid. All potential energies are with respect to $z = 0$ as the datum plane. Hence we get:

$$\frac{d}{dt} \Phi_{\text{tot}} = -\left(\frac{1}{2} v_2^2 + g z_2 \right) w_2$$

First we get the total potential energy in the fluid. This is done by integrating $\rho g z$ over the volume (regarded as a truncated cone):

$$\Phi_{\text{tot}} = \int_{V(t)} \rho \hat{\Phi} dV = \int_{V(t)} \rho g z dV = \rho g \int_{z_2}^z \bar{z} \cdot \pi \bar{r}^2 d\bar{z}$$

$$= \rho g \int_{z_2}^z \bar{z} \cdot \pi \left(\frac{r_2}{z_2} \bar{z} \right)^2 d\bar{z} = \rho g \pi \left(\frac{r_2}{z_2} \right)^2 \int_{z_2}^z \bar{z}^3 d\bar{z} = \frac{\rho g \pi}{4} \left(\frac{r_2}{z_2} \right)^2 (z^4 - z_2^4)$$

Then the unsteady mechanical energy balance becomes

$$\frac{\rho g \pi}{4} \left(\frac{r_2}{z_2} \right)^2 \frac{d}{dt} (z^4 - z_2^4) = - \left(\frac{1}{2} v_2^2 + g z_2 \right) (\rho v_2 \pi r_2^2) \quad \text{or}$$

$$\rho g \pi \left(\frac{r_2}{z_2} \right)^2 z^3 \frac{dz}{dt} = - \left(\frac{1}{2} v_2^2 + g z_2 \right) (\rho v_2 \pi r_2^2)$$

Then, using the result of the mass balance to eliminate the time derivative, we get

$$\rho g \pi \left(\frac{r_2}{z_2} \right)^2 z^3 \left(- \frac{z_2^2}{z^2} v_2 \right) = - \left(\frac{1}{2} v_2^2 + g z_2 \right) (\rho v_2 \pi r_2^2)$$

After dividing through by $-\rho \pi r_2^2 v_2$ we get Torricelli's law:

$$g z = \frac{1}{2} v_2^2 + g z_2 \quad \text{or} \quad v_2 = \sqrt{2g(z - z_2)}$$

c. Using the results of (a) and (b) we get

$$- \frac{dz}{dt} = \left(\frac{z_2}{z} \right)^2 \sqrt{2g(z - z_2)} \approx \frac{\sqrt{2g z_2^2}}{z^{3/2}}$$

By making the indicated simplification, we can perform the integration, along with the initial condition that at $t = 0$, $z = z_0$, to get

$$z_0^{5/2} - z^{5/2} = \frac{5}{2} \sqrt{2g z_2^2} t \quad \text{and} \quad t_{\text{efflux}} = \frac{1}{5} \left(\frac{z_0}{z_2} \right)^2 \sqrt{\frac{2z_0}{g}}$$

To get the efflux time, we have assumed that $z = z_2 \approx 0$ when the container is empty.

7B.11 Disintegration of wood chips

We start with Eq. 7.5-10, taking plane 1 at the top of the slurry dispersion and plane 2 right at the outlet to the digester; we also take plane 2 to be the datum plane for the calculation of the potential energy:

$$\frac{1}{2}(v_2^2 - 0) + (0 - gz_1) + \frac{1}{\rho}(p_2 - p_1) = 0$$

Therefore the exit velocity is

$$\begin{aligned} v_2 &= \sqrt{2\left(\frac{1}{\rho}(p_1 - p_2) + gz_1\right)} \\ &= \sqrt{2\left(\frac{(100 \text{ lb}_f/\text{in}^2)(144 \text{ in}^2/\text{ft}^2)(32.2 \text{ poundals}/\text{lb}_f)}{65 \text{ lb}_m/\text{ft}^3} + (32.2 \text{ ft}/\text{s}^2)(20 \text{ ft})\right)} \\ &= \sqrt{15550} = 124 \text{ ft/s} \end{aligned}$$

Therefore the mass rate of flow at the exit is

$$w_2 = \rho v_2 S_2 = (65 \text{ lb}_m/\text{ft}^3)(124 \text{ ft/s})\left(\frac{1}{9}\pi \text{ ft}^2\right) = 2910 \text{ lb}_m/\text{s}$$

Next, we apply Eq. 7.2-3 (the momentum balance) to get the impact force (assuming that the pressure and external force terms are quite negligible):

$$\begin{aligned} F_{f \rightarrow s} &= v_2 w_2 = (124 \text{ ft/s})(2910 \text{ lb}_m/\text{s})(1/32 \text{ lb}_f/\text{poundals}) \\ &= 10,900 \text{ lb}_f \end{aligned}$$

7B.12 Criterion for Vapor-Free Flow in a Piping System.

In the piping system of Fig. 7.5-1, the criterion $p > p_{\text{vap}}$ might be violated either (i) in the pump, or (ii) at a plane "A" just downstream of the final elbow. Lacking data on the NPSH (net positive suction head) requirements of the pump at the given operating condition, we can only test for condition (ii); further information on (i) is available in *Perry's Chemical Engineers' Handbook* and in unit operations texts.

Applying Eq. 7.5-10 from plane "A" to just above plane "2", we find (since $v_A = v_2$ and no fittings or enlargement loss occurs in this vertical section),

$$g(z_2 - z_A) + \frac{p_2 - p_A}{\rho} = -\frac{v_2^2}{2} \frac{L_{A \rightarrow 2}}{R_h} f$$

which indicates the minimum pressure to be

$$p_A = p_2 + \rho \left[g(z_2 - z_A) + \frac{v_2^2}{2} \frac{L_{A \rightarrow 2}}{R_h} f \right]$$

Using values from page 208 and setting $p_A = 1 \text{ atm} = 6.8087 \times 10^4 \text{ poundals/ft}^2$, we find the pressure at plane "A" to be

$$\begin{aligned} p_A &= 68087 + 62.4 \left[32.2(-20) + \frac{20}{5 + 300 + 100 + 120} (85) \right] \text{ poundals/ft}^2 \\ &= 68087 - 40186 + 3 = 27905 \text{ poundals/ft}^2 \\ &= 0.41 \text{ atm} \end{aligned}$$

Since the minimum pressure is well above the vapor pressure of water at the system conditions, the pipe will run full if the pump does. For mixtures, one must use the bubble-point pressure rather than the vapor pressure.

7C.1 End corrections in tube viscometers.

We apply the steady-state mechanical energy balance, in the form of Eq. 7.4-7, with the assumptions of incompressible fluid and no mechanical work:

$$\frac{1}{2} \left(\frac{\langle v_1^3 \rangle}{\langle v_1 \rangle} - \frac{\langle v_2^3 \rangle}{\langle v_2 \rangle} \right) + g(h_1 - h_2) + \frac{1}{\rho} (p_1 - p_2) = \hat{E}_v$$

where "1" and "2" are general labels for the input and output streams. We label the plane at the bottom of the tanks as Plane 5, and designate by p_{atm} the pressure at the outlet plane.

Then for Run A, we apply the balance to the region between Plane 5 and Plane 2:

$$\frac{1}{2} \left(\frac{\langle v_5^3 \rangle}{\langle v_5 \rangle} - \frac{\langle v_2^3 \rangle}{\langle v_2 \rangle} \right) + gL_A + \frac{1}{\rho} (p_A + \rho g l_A - p_{\text{atm}}) = \hat{E}_v (5 \rightarrow 2)$$

Note that the kinetic energy term is nonzero, because the velocity distribution at the inlet and outlet are not the same.

For Run B, we apply the balance to the region between Plane 5 and Plane 0 to get:

$$\frac{1}{2} \left(\frac{\langle v_5^3 \rangle}{\langle v_5 \rangle} - \frac{\langle v_0^3 \rangle}{\langle v_0 \rangle} \right) + gL_A + \frac{1}{\rho} (p_B + \rho g l_B - p_0) = \hat{E}_v (5 \rightarrow 0)$$

We now subtract the second equation from the first, noting that the kinetic energy terms will exactly cancel, as will the viscous dissipation terms, since the flow rates (and hence the velocity profiles) in the two systems are equal. This gives

$$\frac{1}{\rho} [(p_A - p_B) + \rho g (l_A - l_B) + (p_0 - p_{\text{atm}})] = 0 \quad (*)$$

Next we apply the mechanical energy balance to the region between Plane 0 and Plane 4 in Run B:

$$g(L_B - L_A) + \frac{1}{\rho}(p_0 - p_{\text{atm}}) = \hat{E}_v(0 \rightarrow 4)$$

Here the kinetic energy terms do cancel, because the flow is fully developed at the inlet and outlet planes. The dissipation term can now be calculated by using Eq. 7.5-7 together with Eq. 2.3-22:

$$\hat{E}_v(0 \rightarrow 4) = \frac{F_{f \rightarrow s}(0 \rightarrow 4)}{\rho(\pi R^2)} = \frac{1}{\rho}[(p_0 - p_{\text{atm}}) + g(L_B - L_A)]$$

But, according to Fn. 1 on p 50,

$$\mathfrak{P}_0 - \mathfrak{P}_4 = (p_0 - p_4) + \rho g(h_0 - h_4) = (p_0 - p_{\text{atm}}) + \rho g(L_B - L_A)$$

Therefore, by combining the last three equations we get

$$g(L_B - L_A) + \frac{1}{\rho}(p_0 - p_{\text{atm}}) = \frac{1}{\rho}(\mathfrak{P}_0 - \mathfrak{P}_4) \quad (**)$$

Now Eq. (**) can be rearranged and then combined with Eq. (*) to give:

$$\begin{aligned} \frac{\mathfrak{P}_0 - \mathfrak{P}_4}{L_B - L_A} &= \rho g + \frac{p_0 - p_{\text{atm}}}{L_B - L_A} \\ &= \rho g - \frac{1}{L_B - L_A} [(p_A - p_B) + \rho g(l_B - l_A)] \\ &= \frac{p_B - p_A}{L_B - L_A} + \rho g \left(1 - \frac{l_A - l_B}{L_B - L_A} \right) \\ &= \frac{p_B - p_A}{L_B - L_A} + \rho g \left(1 + \frac{l_B - l_A}{L_B - L_A} \right) \end{aligned}$$

in agreement with Eq. 7C.1-1.

7D.1 Derivation of the macroscopic balances from the equations of change

a. Derivation of the macroscopic mass balance

We start by integrating the equation of continuity, Eq.3.1-4 over the macroscopic flow system of Fig. 7.0-1:

$$\int_{V(t)} \frac{\partial \rho}{\partial t} dV = - \int_{V(t)} (\nabla \cdot \rho \mathbf{v}) dV$$

We write $V(t)$ to remind ourselves that the volume may be changing with time because of the presence of moving parts within the system. We now apply the Leibniz formula (Eq. A.5-5) to the left side and the Gauss divergence theorem (Eq. A.5-2) to the right side to get

$$\frac{d}{dt} \int_{V(t)} \rho dV - \int_{S(t)} \rho (\mathbf{n} \cdot \mathbf{v}_S) dS = - \int_{S(t)} (\mathbf{n} \cdot \rho \mathbf{v}) dS$$

We now combine the two surface integrals thus:

$$\frac{d}{dt} m_{\text{tot}} = - \int_{S(t)} (\mathbf{n} \cdot \rho (\mathbf{v} - \mathbf{v}_S)) dS$$

Next we divide the surface up into four parts as indicated on p. 221, right after Eq. 7.8-2. We also introduce the assumptions (i) and (ii) listed in §§7.1 and 7.2. Then we divide the surface into four nonoverlapping parts, $S = S_1 + S_2 + S_f + S_m$, and write the right side of the above equation as the sum of four contributions:

$$\begin{aligned} \frac{d}{dt} m_{\text{tot}} = & - \int_{S_1} (\mathbf{n} \cdot \rho (\mathbf{v} - \mathbf{v}_S)) dS - \int_{S_2} (\mathbf{n} \cdot \rho (\mathbf{v} - \mathbf{v}_S)) dS - \int_{S_f} (\mathbf{n} \cdot \rho (\mathbf{v} - \mathbf{v}_S)) dS \\ & - \int_{S_m} (\mathbf{n} \cdot \rho (\mathbf{v} - \mathbf{v}_S)) dS \end{aligned}$$

We now evaluate seriatim the terms on the right side:

The surface S_1 is the inlet plane, which is not moving so that $\mathbf{v}_S = 0$, and the outwardly directed unit normal vector \mathbf{n} is the negative of the vector \mathbf{u}_1 , which indicates the direction of the flow. Since $\mathbf{v} = \mathbf{v}_1$ at this surface is assumed (assumption (i)) to be exactly in the

direction of flow, we may say that $\mathbf{v} = \mathbf{u}_1 v_1$. The evaluation of the integral proceeds as follows:

$$\int_{S_1} (\mathbf{n} \cdot \rho(\mathbf{v} - \mathbf{v}_s)) dS = + \int_{S_1} (\mathbf{u}_1 \cdot \mathbf{u}_1 v_1 \rho) dS = + \int_{S_1} \rho v_1 dS = \rho_1 \langle v_1 \rangle S_1 = w_1$$

where it has been assumed that the density is constant over the cross section (assumption (ii)). The integral over the exit plane is evaluated in the same way, the only difference being that \mathbf{n} is the same as \mathbf{u}_2 , so that

$$- \int_{S_2} (\mathbf{n} \cdot \rho(\mathbf{v} - \mathbf{v}_s)) dS = - \int_{S_2} (\mathbf{u}_2 \cdot \mathbf{u}_2 v_2 \rho) dS = + \int_{S_2} \rho v_2 dS = \rho_2 \langle v_2 \rangle S_2 = w_2$$

On the fixed surfaces both \mathbf{v} and \mathbf{v}_s are zero, so that $S_f = 0$. Also, on the moving surfaces, $\mathbf{v} = \mathbf{v}_s$, with the result that $S_m = 0$. Therefore, Eq. 7.1-1 follows at once:

$$\frac{d}{dt} m_{\text{tot}} = \rho_1 \langle v_1 \rangle S_1 - \rho_2 \langle v_2 \rangle S_2$$

and the definitions of w_1 and w_2 lead to Eq. 7.1-2.

b. Derivation of the macroscopic momentum balance

First we integrate Eq. 3.2-9 over the volume of the system:

$$\int_{V(t)} \left(\frac{\partial}{\partial t} \rho \mathbf{v} \right) dV = - \int_{V(t)} (\nabla \cdot \rho \mathbf{v} \mathbf{v}) dV - \int_{V(t)} (\nabla p) dV - \int_{V(t)} [\nabla \cdot \boldsymbol{\tau}] dV + \int_{V(t)} \rho \mathbf{g} dV$$

We now manipulate seriatim the terms in this equation so that they can be interpreted. In so doing we make use of assumptions i-iv in §§7.1 and 2, and follow the procedures in (a) above. The first term is rearranged by using the Leibniz formula (as applied to a vector function), and then \mathbf{P}_{tot} is introduced as the definition of the integral over the momentum per unit volume, and the second term is rearranged, thus:

$$\int_{V(t)} \left(\frac{\partial}{\partial t} \rho \mathbf{v} \right) dV = \frac{d}{dt} \int_{V(t)} \rho \mathbf{v} dV - \int_{S(t)} \rho \mathbf{v} (\mathbf{v}_s \cdot \mathbf{n}) dS = \frac{d}{dt} \mathbf{P}_{\text{tot}} - \int_{S(t)} [\mathbf{n} \cdot \rho \mathbf{v} \mathbf{v}_s] dS$$

The first term and third terms on the right of the momentum balance are treated by using Eq. A.5-3, and the pressure term is treated by using Eq. A.5-2:

$$- \int_{V(t)} (\nabla \cdot \rho \mathbf{v} \mathbf{v}) dV = - \int_{S(t)} [\mathbf{n} \cdot \rho \mathbf{v} \mathbf{v}] dS$$

$$- \int_{V(t)} [\nabla \cdot \boldsymbol{\tau}] dV = - \int_{S(t)} [\mathbf{n} \cdot \boldsymbol{\tau}] dS$$

$$- \int_{V(t)} (\nabla p) dV = - \int_{S(t)} \mathbf{n} p dS$$

Finally, the last term is integrated (with \mathbf{g} constant) and the definition of the total mass within the system (given above):

$$+ \int_{V(t)} \rho \mathbf{g} dV = + m_{\text{tot}} \mathbf{g}$$

When the above contributions to the momentum balance are combined we get the following equation:

$$\frac{d}{dt} \mathbf{P}_{\text{tot}} = - \int_{S(t)} [\mathbf{n} \cdot \rho \mathbf{v} (\mathbf{v} - \mathbf{v}_s)] dS - \int_{S(t)} \mathbf{n} p dS - \int_{S(t)} [\mathbf{n} \cdot \boldsymbol{\tau}] dS + m_{\text{tot}} \mathbf{g}$$

The surface integrals are interpreted by partitioning the surface of the system as before: $S = S_1 + S_2 + S_f + S_m$. The first integral on the right is zero on the fixed and moving surfaces, and the integrations on the inlet and outlet surfaces give the following:

$$\begin{aligned} - \int_{S(t)} [\mathbf{n} \cdot \rho \mathbf{v} (\mathbf{v} - \mathbf{v}_s)] dS &= - \int_{S_1} [-\mathbf{u}_1 \cdot \rho (\mathbf{u}_1 v_1) (\mathbf{u}_1 v_1)] - \int_{S_2} [\mathbf{u}_2 \cdot \rho (\mathbf{u}_2 v_2) (\mathbf{u}_2 v_2)] \\ &= +\mathbf{u}_1 \rho_1 \int_{S_1} v_1^2 dS - \mathbf{u}_2 \rho_2 \int_{S_2} v_2^2 dS \\ &= +\mathbf{u}_1 \rho_1 \langle v_1^2 \rangle S_1 - \mathbf{u}_2 \rho_2 \langle v_2^2 \rangle S_2 \end{aligned}$$

where assumption (ii) has been used--that the density of the fluid is constant over the cross-section. Similarly the contributions of the pressure integral at the entry and exit planes are:

$$- \int_{S_1+S_2} \mathbf{n} p dS = +\mathbf{u}_1 p_1 S_1 - \mathbf{u}_2 p_2 S_2$$

where, according to assumption (iv), the pressure is assumed constant over the cross-section. The integral of the pressure over the fixed and moving surfaces contributes to the force of the fluid on the solid surfaces, and the negative of that is the pressure force of the solid on the fluid: $\mathbf{F}_{s \rightarrow f}^{(p)}$. Similarly the integral over the $[\mathbf{n} \cdot \boldsymbol{\tau}]$ term over the fixed and moving surfaces gives a contribution $\mathbf{F}_{s \rightarrow f}^{(\tau)}$. Therefore

$$- \int_{S_1+S_2} \mathbf{n} p dS - \int_{S_1+S_2} [\mathbf{n} \cdot \boldsymbol{\tau}] dS = \mathbf{F}_{s \rightarrow f}^{(p)} + \mathbf{F}_{s \rightarrow f}^{(\tau)} = \mathbf{F}_{s \rightarrow f}$$

According to assumption (iii) we neglect the contribution of the $[\mathbf{n} \cdot \boldsymbol{\tau}]$ terms at the entry and exit planes. When all of the contributions above are collected, we get Eq. 7.2-1.

$$\frac{d}{dt} \mathbf{P}_{\text{tot}} = +\rho_1 \langle v_1^2 \rangle S_1 \mathbf{u}_1 - \rho_2 \langle v_2^2 \rangle S_2 \mathbf{u}_2 + p_1 S_1 \mathbf{u}_1 - p_2 S_2 \mathbf{u}_2 + \mathbf{F}_{s \rightarrow f} + m_{\text{tot}} \mathbf{g}$$

and the definitions of w_1 and w_2 then lead to Eq. 7.2-2.

8A.1 Flow of a polyisoprene solution in a pipe

We use Eq. 8.3-9, for the mass flow rate of a power-law fluid in a circular pipe:

$$w = \frac{\pi R^3 \rho}{(1/n) + 3} \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L) R}{2mL} \right)^{1/n}$$

Consider two systems, labeled I and II differing only in the radius R and the length L . Then, for the same mass flow rate and the same pressure difference, and also the same values of the density and the power-law parameters, from the above equation we know that the dimensions of the two systems will be related by

$$\frac{R_I^{3+(1/n)}}{L_I^{1/n}} = \frac{R_{II}^{3+(1/n)}}{L_{II}^{1/n}}$$

Solving for the radius of the second system, we get

$$R_{II} = R_I \left(\frac{L_{II}}{L_I} \right)^{1/(3n+1)}$$

Inserting the numerical values for this problem gives

$$R_{II} = \left(\frac{1.3 \text{ cm}}{2} \right) \left(\frac{30.6 \text{ m}}{10.2 \text{ m}} \right)^{5/8} = 1.25 \text{ cm}$$

8A.2 Pumping of a polyethylene oxide solution

The starting point is the same as for Problem 8A.1. The mass rate of flow must be the same for the two designs, so that

$$\frac{\pi R_1^3 \rho}{(1/n) + 3} \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L) R_1}{2mL} \right)^{1/n} = 2 \cdot \frac{\pi R_2^3 \rho}{(1/n) + 3} \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L) R_2}{2mL} \right)^{1/n}$$

Therefore

$$R_1^{3+(1/n)} = 2R_2^{3+(1/n)}$$

whence

$$\left(\frac{R_2}{R_1} \right)^{3+(1/n)} = \frac{1}{2}$$

Therefore

$$\frac{D_2}{D_1} = \frac{R_2}{R_1} = \left(\frac{1}{2} \right)^{n/(3n+1)}$$

and, for $n = 0.6$

$$D_2 = D_1 \left(\frac{1}{2} \right)^{0.6/2.8} = (27 \text{ cm}) \left(\frac{1}{2} \right)^{0.2143} = 23.3 \text{ cm}$$

8B.1 Flow of a polymeric film

a. The momentum flux distribution is the same as Eq. 2.2-13, inasmuch as this result depends only on momentum conservation, and not on the constitutive relation.

b. Next we write the power-law model as

$$\tau_{xz} = -\eta \frac{dv_z}{dx} = -m\dot{\gamma}^{n-1} \frac{dv_z}{dx} = -m \left(-\frac{dv_z}{dx} \right)^{n-1} \frac{dv_z}{dx} = m \left(-\frac{dv_z}{dx} \right)^n$$

which is the appropriate form, because dv_z/dx is negative.

c. When the above expression for τ_{xz} is substituted into Eq. 2.2-13, we get a differential equation for the velocity

$$m \left(-\frac{dv_z}{dx} \right)^n = (\rho g \cos \beta) x \quad \text{or} \quad -\frac{dv_z}{dx} = \left(\frac{\rho g \cos \beta}{m} \right)^{1/n} x^{1/n}$$

This equation is integrated to give

$$-v_z = \left(\frac{\rho g \cos \beta}{m} \right)^{1/n} \frac{x^{(1/n)+1}}{(1/n)+1} + C_2$$

Application of the no-slip condition at the wall gives

$$0 = \left(\frac{\rho g \cos \beta}{m} \right)^{1/n} \frac{\delta^{(1/n)+1}}{(1/n)+1} + C_2$$

Subtraction of the last two equations gives

$$\begin{aligned} v_z &= \left(\frac{\rho g \cos \beta}{m} \right)^{1/n} \frac{\delta^{(1/n)+1} - x^{(1/n)+1}}{(1/n)+1} \\ &= \left(\frac{\rho g \delta \cos \beta}{m} \right)^{1/n} \frac{\delta}{(1/n)+1} \left[1 - \left(\frac{x}{\delta} \right)^{(1/n)+1} \right] \end{aligned}$$

d. Then the mass rate of flow in the film is

$$\begin{aligned}
 w &= \int_0^W \int_0^\delta \rho v_z dx dy = \rho W \left(\frac{\rho g \delta \cos \beta}{m} \right)^{1/n} \frac{\delta^2}{(1/n)+1} \int_0^1 (1 - \xi^{(1/n)+1}) d\xi \\
 &= \rho W \left(\frac{\rho g \delta \cos \beta}{m} \right)^{1/n} \frac{\delta^2}{(1/n)+2} \xrightarrow{n \rightarrow 1, m \rightarrow \mu} \frac{\rho^2 g W \delta^3 \cos \beta}{3\mu}
 \end{aligned}$$

8B.2 Power-law flow in a narrow slit

Substituting Eq. 8.3-11 into Eq. 2B.3-1, we get for $-B \leq x \leq 0$

$$-m \left(\frac{dv_z}{dx} \right)^n = \frac{(\mathcal{P}_0 - \mathcal{P}_L)x}{2L} \quad \text{or} \quad \frac{dv_z}{d(-x)} = - \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{2mL} \right)^{1/n} (-x)^{1/n}$$

Integration then gives

$$\begin{aligned} v_z &= \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{2mL} \right)^{1/n} \frac{B^{(1/n)+1} - (-x)^{(1/n)+1}}{(1/n)+1} \\ &= \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)B}{2mL} \right)^{1/n} \frac{B}{(1/n)+1} \left[1 - \left(\frac{-x}{B} \right)^{(1/n)+1} \right] \quad (-B \leq x \leq 0) \end{aligned}$$

This may be combined with Eq. 8.3-13 to get a solution that is valid for $-B \leq x \leq B$

$$v_z = \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)B}{2mL} \right)^{1/n} \frac{B}{(1/n)+1} \left[1 - \left| \frac{x}{B} \right|^{(1/n)+1} \right]$$

8B.3 Non-Newtonian flow in an annulus

a. From the equation of motion, the momentum-flux distribution is

$$\frac{1}{r} \frac{d}{dr} (r \tau_{rz}) = 0 \quad \text{or} \quad \tau_{rz} = \frac{C_1}{r}$$

The appropriate expression for the momentum flux component is

$$\tau_{rz} = -m \dot{\gamma}^{n-1} \frac{dv_z}{dr} = -m \left(-\frac{dv_z}{dr} \right)^{n-1} \frac{dv_z}{dr} = +m \left(-\frac{dv_z}{dr} \right)^n$$

When the above two expressions are combined, one gets a differential equation for the velocity distribution, which can be integrated to give (after applying the boundary conditions)

$$\frac{v_z}{v_0} = \frac{\xi^{1-(1/n)} - 1}{\kappa^{1-(1/n)} - 1} \quad (\text{here } \xi = r/R)$$

b. In the limit that $n \rightarrow 1$ and $m \rightarrow \mu$, we have to use L'Hôpital's rule

$$\frac{v_z}{v_0} = \lim_{\varepsilon \rightarrow 0} \frac{\xi^\varepsilon - 1}{\kappa^\varepsilon - 1} = \lim_{\varepsilon \rightarrow 0} \frac{\xi^\varepsilon \ln \xi}{\kappa^\varepsilon \ln \kappa} = \frac{\ln \xi}{\ln \kappa} \quad (\text{see Problem 2B.7})$$

c. The mass flow rate is then

$$\begin{aligned} w &= 2\pi \int_{\kappa R}^R \rho v_z r dr = 2\pi R^2 \rho v_0 \int_{\kappa}^1 \left(\frac{\xi^{2-(1/n)} - \xi}{\kappa^{1-(1/n)} - 1} \right) d\xi \\ &= \frac{2\pi R^2 \rho v_0}{\kappa^{1-(1/n)} - 1} \left(\frac{1 - \kappa^{3-(1/n)}}{3 - (1/n)} - \frac{1 - \kappa^2}{2} \right) \quad (n \neq \frac{1}{3}) \end{aligned}$$

d. When $n = \frac{1}{3}$ use of L'Hôpital's rule gives

$$w = \frac{2\pi \kappa^2 R^2 \rho v_0}{1 - \kappa^2} \left(\ln \frac{1}{\kappa} - \frac{1 - \kappa^2}{2} \right)$$

e. When L'Hôpital's rule is used we get

$$w = \pi R^2 \rho v_0 \left(\frac{1 - \kappa^2}{2 \ln(1/\kappa)} - \kappa^2 \right)$$

8B.4 Flow of a polymeric liquid in a tapered tube

We consider a small region of the tapered tube to be a straight tube over a short distance dz ; then we can write "locally"

$$w = \frac{\pi R^3 \rho}{(1/n) + 3} \left[-\frac{d\mathcal{P}}{dz} \frac{R}{2m} \right]^{1/n}$$

Take the n th power of both sides to get

$$-\frac{d\mathcal{P}}{dz} = \frac{2m}{R} \left[\frac{w}{\pi R^3 \rho} \left(\frac{1}{n} + 3 \right) \right]^n$$

in which R is a function of z :

$$R = R_0 + \left(\frac{R_L - R_0}{L} \right) z$$

It is easier to integrate the differential equation if we rewrite it as

$$-\frac{d\mathcal{P}}{dR} \frac{dR}{dz} = -\frac{d\mathcal{P}}{dR} \left(\frac{R_L - R_0}{L} \right) = \frac{2m}{R} \left[\frac{w}{\pi R^3 \rho} \left(\frac{1}{n} + 3 \right) \right]^n$$

Then when this equation is integrated with respect to R , we get

$$-\int_{\mathcal{P}_0}^{\mathcal{P}_L} d\mathcal{P} = \left(\frac{2mL}{R_L - R_0} \right) \left[\frac{w}{\pi \rho} \left(\frac{1}{n} + 3 \right) \right]^n \int_{R_0}^{R_L} \frac{1}{R^{3n+1}} dR$$

Therefore

$$\begin{aligned} \mathcal{P}_0 - \mathcal{P}_L &= \left(\frac{2mL}{R_L - R_0} \right) \left[\frac{w}{\pi \rho} \left(\frac{1}{n} + 3 \right) \right]^n \left(\frac{R_L^{-3n} - R_0^{-3n}}{-3n} \right) \\ &= \left(\frac{2mL}{3n} \right) \left[\frac{w}{\pi \rho} \left(\frac{1}{n} + 3 \right) \right]^n \left(\frac{R_L^{-3n} - R_0^{-3n}}{R_0 - R_L} \right) \end{aligned}$$

This is the power-law analog of Eq. 2B.10-3.

8B.5 Slit flow of a Bingham fluid

a. For $|x| \leq x_0$ (i.e., in the region where the yield stress is not exceeded), $\eta = \infty$ (according to the upper equation of Eq. 8B.5-1. But the expression for the shear stress is, from Eq. 8.3-2: $\tau_{xz} = -\eta dv_z/dx$. Since the shear stress is finite, the velocity gradient must be equal to zero. This is the plug-flow region.

For $|x| \geq x_0$ (i.e., in the region where the yield stress is exceeded), the lower equation of Eq. 8B.5-1 has to be used. This means that in the region where $x \geq x_0$, the velocity will be decreasing in the positive x direction, so that $\gamma = -dv_z/dx$ is required so that γ will be positive. Similarly, when $x \leq -x_0$, the velocity will be increasing in the positive x direction, so that $\gamma = +dv_z/dx$ is needed in order to guarantee that γ be positive. Hence we have:

$$\tau_{xz} = -\mu_0 \frac{dv_z}{dx} - \tau_0 \quad \text{for} \quad -B \leq x \leq -x_0$$

$$\tau_{xz} = -\mu_0 \frac{dv_z}{dx} + \tau_0 \quad \text{for} \quad +x_0 \leq x \leq +B$$

Since the flow will be symmetric about the plane $x=0$, we need solve for the velocity distribution only in one half of the slit. We choose to work with the region $0 \leq x \leq +B$.

b. Substitution of the upper relation of Eq. 8B.5-3 into Eq. 2B.3-1 gives

$$-\mu_0 \frac{dv_z}{dx} + \tau_0 = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} x \quad \text{or} \quad \frac{dv_z}{dx} = -\frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu_0 L} x + \frac{\tau_0}{\mu_0}$$

Integration gives:

$$v_z = -\frac{\mathcal{P}_0 - \mathcal{P}_L}{2\mu_0 L} x^2 + \frac{\tau_0}{\mu_0} x + C$$

Applying the no-slip boundary condition at the slit wall ($z = B$), we get

$$0 = -\frac{\mathfrak{P}_0 - \mathfrak{P}_L}{2\mu_0 L} B^2 + \frac{\tau_0}{\mu_0} B + C$$

Subtracting these last two equations eliminates the integration constant and gives

$$v_z = \frac{(\mathfrak{P}_0 - \mathfrak{P}_L)B^2}{2\mu_0 L} \left[1 - \left(\frac{x}{B} \right)^2 \right] - \frac{\tau_0 B}{\mu_0} \left(1 - \frac{x}{B} \right) \quad (+x_0 \leq x \leq +B)$$

$$v_z = \frac{(\mathfrak{P}_0 - \mathfrak{P}_L)B^2}{2\mu_0 L} \left[1 - \left(\frac{x_0}{B} \right)^2 \right] - \frac{\tau_0 B}{\mu_0} \left(1 - \frac{x_0}{B} \right) \quad (0 \leq x \leq x_0)$$

c. The mass flow rate is then obtained from

$$w = W\rho \int_{-B}^{+B} v_z dx = 2 \cdot W\rho \int_0^{+B} v_z dx$$

One could split this integral up into two parts, one for the region $0 \leq x \leq x_0$ and the other for $+x_0 \leq x \leq +B$, and evaluate them using the velocity profiles worked out in (b). A better way to do the integral is to integrate by parts:

$$w = 2 \cdot W\rho \left[v_z x \Big|_0^B - \int_0^B x \left(\frac{dv_z}{dx} \right) dx \right]$$

Then the first term is zero at both limits, and the second term has no contribution from the portion from x to x_0 . Hence

$$\begin{aligned} w &= 2 \cdot W\rho \int_{x_0}^B x \left(-\frac{dv_z}{dx} \right) dx = 2 \cdot W\rho \int_{x_0}^B x \left(\frac{\mathfrak{P}_0 - \mathfrak{P}_L}{\mu_0 L} x - \frac{\tau_0}{\mu_0} \right) dx \\ &= 2 \cdot W\rho \left[\frac{(\mathfrak{P}_0 - \mathfrak{P}_L)x^3}{3\mu_0 L} - \frac{\tau_0 x^2}{2\mu_0} \right]_{x_0}^B \\ &= 2 \cdot W\rho \left[\frac{(\mathfrak{P}_0 - \mathfrak{P}_L)B^3}{3\mu_0 L} - \frac{\tau_0 B^2}{2\mu_0} - \frac{(\mathfrak{P}_0 - \mathfrak{P}_L)x_0^3}{3\mu_0 L} + \frac{\tau_0 x_0^2}{2\mu_0} \right] \end{aligned}$$

$$= \frac{2(\mathcal{P}_0 - \mathcal{P}_L)B^3W\rho}{3\mu_0L} \left[1 - \frac{3}{2} \left(\frac{\tau_0L}{(\mathcal{P}_0 - \mathcal{P}_L)B} \right) + \frac{1}{2} \left(\frac{\tau_0L}{(\mathcal{P}_0 - \mathcal{P}_L)B} \right)^3 \right]$$

In going from the next-to-last step to the last step of this development, we have used the defining equation for x_0 , given in Eq. 8B.5-2, to eliminate x_0 in favor of the yield stress.

When the yield stress goes to zero, The above result simplifies to the expression given in the answer to part (c) of Problem 2B.3 for the Newtonian fluid.

8B.6 Derivation of the Buckingham-Reiner Equation

The equation of motion for tube flow can be integrated to give $\tau_{rz} = \tau_R(r/R)$, where $\tau_R = (\mathcal{P}_0 - \mathcal{P}_L)R/2L$ (cf. Eq. 2.3-13). The outer boundary of the plug-flow region is $r_0 = (\tau_0/\tau_R)R$.

The Bingham fluid (see Problem 8B.5) gives for the shear stress component

$$\tau_{rz} = -\mu_0 \frac{dv_z}{dr} + \tau_0 \quad (r \geq r_0) \quad \text{and} \quad \frac{dv_z}{dr} = 0 \quad (r \leq r_0)$$

In the region ($r \geq r_0$), we can combine the shear-stress expression with the Bingham formula to get

$$-\mu_0 \frac{dv_z}{dr} + \tau_0 = \tau_R \frac{r}{R}$$

Integration gives

$$v_z = -\frac{\tau_R}{2\mu_0} r^2 + \frac{\tau_0}{\mu_0} r + C$$

The integration constant is determined from the boundary condition that the velocity vanishes at the tube wall (the "no slip condition"). Then we get

$$v_z = \frac{\tau_R R}{2\mu_0} \left[1 - \left(\frac{r}{R} \right)^2 \right] - \frac{\tau_0 R}{\mu_0} \left[1 - \left(\frac{r}{R} \right) \right] \quad (r \geq r_0)$$

Since the velocity must be continuous at $r = r_0$, we can set $r = r_0$ in this last expression and get

$$\begin{aligned} v_z &= \frac{\tau_R R}{2\mu_0} \left[1 - \left(\frac{r_0}{R} \right)^2 \right] - \frac{(r_0 \tau_R / R) R}{\mu_0} \left[1 - \left(\frac{r_0}{R} \right) \right] \\ &= \frac{\tau_R R}{2\mu_0} \left[1 - \left(\frac{r_0}{R} \right)^2 - 2 \left(\frac{r_0}{R} \right) + 2 \left(\frac{r_0}{R} \right)^2 \right] = \frac{\tau_R R}{2\mu_0} \left[1 - \left(\frac{r_0}{R} \right) \right]^2 \quad (r \leq r_0) \end{aligned}$$

The mass rate of flow is then

$$w = \int_0^{2\pi} \int_0^R \rho v_z r dr d\theta = 2\pi\rho \int_0^R v_z r dr$$

We then integrate by parts (cf. Problem 8B.5) to get

$$w = 2\pi\rho \left[\frac{1}{2} r^2 v_z \Big|_0^R - \frac{1}{2} \int_0^R r^2 \frac{dv_z}{dr} dr \right]$$

The first term is zero at both limits, and the second contributes nothing in the range ($r \leq r_0$). Therefore we get

$$\begin{aligned} w &= \pi\rho \int_{r_0}^R r^2 \left(\frac{\tau_R}{\mu_0} \frac{r}{R} - \frac{\tau_0}{\mu_0} \right) dr \\ &= \frac{\pi R^3 \tau_R \rho}{4\mu_0} \left[1 - \frac{4}{3} \left(\frac{\tau_0}{\tau_R} \right) + \frac{1}{3} \left(\frac{\tau_0}{\tau_R} \right)^4 \right] \end{aligned}$$

In getting the final expression, r_0 has been eliminated by using the relation $r_0 = (\tau_0/\tau_R)R$. When the yield stress is set equal to zero, this result simplifies to the Hagen-Poiseuille relation for Newtonian fluids.

8B.7 The complex-viscosity components for the Jeffreys fluid

a. The yx component of Eq. 8.4-4 is

$$\tau_{yx} + \lambda_1 \frac{d\tau_{yx}}{dt} = -\eta_0 \left(\dot{\gamma}_{yx} + \lambda_2 \frac{d\dot{\gamma}_{yx}}{dt} \right)$$

Dividing by λ_1 , rearranging, and introducing complex quantities, we get

$$\begin{aligned} \frac{d\tau_{yx}}{dt} + \frac{1}{\lambda_1} \tau_{yx} &= -\frac{\eta_0 \dot{\gamma}^0}{\lambda_1} \left(\Re\{e^{i\omega t}\} + \lambda_2 \frac{d}{dt} \Re\{e^{i\omega t}\} \right) \\ &= -\frac{\eta_0 \dot{\gamma}^0}{\lambda_1} \Re\{e^{i\omega t} (1 + i\lambda_2 \omega)\} \end{aligned}$$

The only reason for using the complex representation of the trigonometric functions is that then we have to integrate only exponentials, and not products of exponentials and trigonometric functions.

Thus, we now have to integrate the above first-order differential equation to which the solution is

$$\tau_{yx} = -\frac{\eta_0 \dot{\gamma}^0}{\lambda_1} e^{-t/\lambda_1} \Re\left\{ \int_{-\infty}^t e^{+t'/\lambda_1} e^{i\omega t'} (1 + i\lambda_2 \omega) dt' + C \right\}$$

If we assume that the fluid is in an unstressed state at $t = -\infty$, then the integration constant C is zero. The integral can now be evaluated to give:

$$\begin{aligned} \tau_{yx} &= -\frac{\eta_0 \dot{\gamma}^0}{\lambda_1} e^{-t/\lambda_1} \Re\left\{ \frac{(1 + i\lambda_2 \omega)}{(1/\lambda_1) + i\omega} e^{+t/\lambda_1} e^{i\omega t} \right\} = -\eta_0 \dot{\gamma}^0 \Re\left\{ \frac{(1 + i\lambda_2 \omega)}{(1 + i\lambda_1 \omega)} e^{i\omega t} \right\} \\ &= -\eta_0 \dot{\gamma}^0 \Re\left\{ \frac{(1 + i\lambda_2 \omega)(1 - i\lambda_1 \omega)}{1 + (\lambda_1 \omega)^2} (\cos \omega t + i \sin \omega t) \right\} \end{aligned}$$

Then we collect the real and imaginary parts within the braces, and discard the imaginary parts to get

$$\tau_{yx} = - \left[\eta_0 \frac{1 + \lambda_1 \lambda_2 \omega^2}{1 + (\lambda_1 \omega)^2} \right] \dot{\gamma}^0 \cos \omega t - \left[\eta_0 \frac{(\lambda_1 - \lambda_2) \omega}{1 + (\lambda_1 \omega)^2} \right] \dot{\gamma}^0 \sin \omega t$$

By comparing this result with the defining equation in Eq. 8.2-4, we see that the quantities in the brackets are $\eta'(\omega)$ and $\eta''(\omega)$ respectively. Equations (F) and (G) of Table 8.5-1 are obtained by setting $\lambda_2 = 0$ and $\lambda_1 = \lambda$. This shows that the Giesekus model is a nonlinear generalization of the linear Maxwell model. On the other hand, the Oldroyd model, as may be seen from Eq. 8.5-12 and 13, is a nonlinear generalization of the Maxwell model.

b. According to fn. 3 on p. 246, the Jeffreys model can be regarded as a superposition of the Newtonian model and the Maxwell model. For the Newtonian fluid (here designated by s) we have

$$\eta'(\omega) = \eta_s \quad \text{and} \quad \eta''(\omega) = 0$$

That is, the Newtonian fluid shows no phase difference between the stress and the velocity gradient. For the Maxwell fluid (here designated by p) we have, by taking just one term in Eqs. 8.4-14 and 15

$$\eta'(\omega) = \frac{\eta_p}{1 + (\lambda_1 \omega)^2} \quad \text{and} \quad \eta''(\omega) = \frac{\eta_p \lambda_1 \omega}{1 + (\lambda_1 \omega)^2}$$

In fn. 3, the connection between the constants of the Jeffreys model and the Newtonian and Maxwell models is given:

$$\eta_0 = \eta_s + \eta_p \quad \text{and} \quad \lambda_2 = \frac{\eta_s}{\eta_s + \eta_p} \lambda_1$$

These relations give η_0 and λ_2 in terms of η_s and η_p . Presently we will need η_s and η_p in terms of η_0 and λ_2 :

$$\eta_s = \frac{\eta_0 \lambda_2}{\lambda_1} \quad \text{and} \quad \eta_p = \frac{\eta_0}{\lambda_1} (\lambda_1 - \lambda_2)$$

We now get the contributions to the complex viscosity for the Jeffreys fluid by summing the contributions of the Newtonian and Maxwell fluids:

$$\begin{aligned}
 \eta'(\omega) &= \eta_s + \frac{\eta_p}{1 + (\lambda_1 \omega)^2} = \frac{\eta_s(1 + \lambda_1 \omega)^2 + \eta_p}{1 + (\lambda_1 \omega)^2} \\
 &= \frac{(\eta_s + \eta_p) + \eta_s(\lambda_1 \omega)^2}{1 + (\lambda_1 \omega)^2} = \frac{\eta_0 + \eta_0 \lambda_1 \lambda_2 \omega^2}{1 + (\lambda_1 \omega)^2} \\
 &= \eta_0 \frac{1 + \lambda_1 \lambda_2 \omega^2}{1 + (\lambda_1 \omega)^2}
 \end{aligned}$$

$$\eta''(\omega) = 0 + \frac{\eta_p \lambda_1 \omega}{1 + (\lambda_1 \omega)^2} = \eta_0 \frac{(\lambda_1 - \lambda_2) \omega}{1 + (\lambda_1 \omega)^2}$$

This is in exact agreement with the results obtained above in (a).

8B.8 Stress relaxation after cessation of shear flow

a. If one writes out the 3-constant Oldroyd model for the special case of steady-state shearing flow, component by component, one gets (cf. Eqs. 8.5-5, 6, 7, and 8):

$$\tau_{yx} + \frac{1}{2}\lambda_1\dot{\gamma}\tau_{xx} - \frac{1}{2}\lambda_1\dot{\gamma}\tau_{yy} = -\eta_0\dot{\gamma}; \quad \tau_{xx} - \lambda_1\dot{\gamma}\tau_{yx} = 0; \quad \tau_{yy} + \lambda_1\dot{\gamma}\tau_{yx} = 0$$

and $\tau_{zz} = 0$. When these equations are solved simultaneously, we get

$$\tau_{yx} = -\eta_0\dot{\gamma} \frac{1}{1+(\lambda_1\dot{\gamma})^2} \quad \text{and} \quad \tau_{xx} = -\eta_0\lambda_1\dot{\gamma}^2 \frac{1}{1+(\lambda_1\dot{\gamma})^2}$$

This expression agrees only somewhat with the data of Fig. 8.2-4. The equation shows the curve for viscosity starting out with the zero-shear-rate viscosity, and then decreasing--which is correct. However, the equation gives $\eta \propto \dot{\gamma}^{-2}$ for high shear rates, and this is nonsense. Then one would have $\tau_{yx} \propto \dot{\gamma}^{-1}$ at high shear rates; this would mean that as one applies more shear to the fluid, the velocity gradient decreases! Therefore the model has to be labeled as unusable at high shear rates. In other words, it can be used only to describe small deviations from Newtonian behavior.

b. For unsteady shear flow, Eq. 8.5-8 describes the shear stress. When the motion is stopped, $\dot{\gamma} = 0$ becomes zero, and the equation gives $\tau_{yx} + \lambda_1(d\tau/dt) = 0$, which can be solved to give $\tau_{yx} = Ce^{-t/\lambda_1}$. The constant is then determined by the requirement that the solution must match at $t = 0$ with the steady state expression in (a). This leads to Eq. 8B.8-2.

c. The normal stress in steady flow was given in (a). Arguments similar to those in (b) lead to

$$\tau_{xx} = -\eta_0\lambda_1\dot{\gamma}^2 \frac{1}{1+(\lambda_1\dot{\gamma})^2} e^{-t/\lambda_1}$$

This predicts (correctly) that the normal stresses also relax after cessation of flow, but experimentally it is found that the normal stresses relax more slowly than the shear stresses.

8B.9 Draining of a tank with an exit pipe

We begin by writing the formula for the mass flow rate through the exit pipe (with radius R_0) for a power-law fluid thus

$$w = \frac{\pi R_0^3 \rho}{(1/n) + 3} \left(-\frac{d\mathcal{P}}{dz} \frac{R_0}{2m} \right)^{1/n} = \frac{\pi R_0^3 \rho}{(1/n) + 3} \left(\frac{\rho g (h + L)}{L} \frac{R_0}{2m} \right)^{1/n}$$

Note that this is a quasi-steady-state approximation. We now equate this to the rate of mass depletion in the tank

$$-\pi R^2 \rho \frac{dh}{dt} = \frac{\pi R_0^3 \rho}{(1/n) + 3} \left(\frac{\rho g (h + L)}{L} \frac{R_0}{2m} \right)^{1/n}$$

We now divide through by $\pi R^2 \rho (h + L)^{1/n}$ and integrate (for $n \neq 1$)

$$-\int_H^0 \frac{dh}{(h + L)^{1/n}} = \frac{R_0^3}{[(1/n) + 3] R^2} \left(\frac{\rho g R_0}{2mL} \right)^{1/n} \int_0^{t_{\text{efflux}}} dt$$

Performing the integration and solving for the efflux time we get

$$t_{\text{efflux}} = \left(\frac{2mL}{\rho g R_0} \right)^{1/n} \frac{[(1/n) + 3] R^2}{R_0^3} \left[\frac{(H + L)^{1-(1/n)} - L^{1-(1/n)}}{1 - (1/n)} \right]$$

To check this result we can let $n \rightarrow 1$ and $m \rightarrow \mu$; we make use of Lhopital's rule to get the result of Problem 7B.9(a):

$$\begin{aligned} t_{\text{efflux}} &= \left(\frac{2\mu L}{\rho g R_0} \right) \frac{4R^2}{R_0^3} \lim_{n \rightarrow 1} \left[\frac{(d/dn) \{ (H + L)^{1-(1/n)} - L^{1-(1/n)} \}}{(d/dn) \{ 1 - (1/n) \}} \right] \\ &= \left(\frac{2\mu L}{\rho g R_0} \right) \frac{4R^2}{R_0^3} \lim_{n \rightarrow 1} \left[\frac{(H + L)^{1-(1/n)} [\ln(H + L)] (1/n^2) - L^{1-(1/n)} [\ln L] (1/n^2)}{(1/n^2)} \right] \\ &= \left(\frac{8\mu L R^2}{\rho g R_0^4} \right) \ln \left(1 + \frac{H}{L} \right) \end{aligned}$$

8B.10 The Giesekus model

a. We have to start by expanding the χ^2 in Eq. (E) of Table 8.5-1 as a function of $(\lambda\dot{\gamma})^2$:

$$\begin{aligned}\chi^2 &= \frac{1 + \frac{1}{2}[16\alpha(1-\alpha)](\lambda\dot{\gamma})^2 - \frac{1}{8}[16\alpha(1-\alpha)]^2(\lambda\dot{\gamma})^4 + \dots - 1}{8\alpha(1-\alpha)(\lambda\dot{\gamma})^2} \\ &= 1 - 4\alpha(1-\alpha)(\lambda\dot{\gamma})^2 + \dots\end{aligned}$$

and then, the series expansion of the square root of this will be

$$\chi = \sqrt{1 - 4\alpha(1-\alpha)(\lambda\dot{\gamma})^2 + \dots} = 1 - 2\alpha(1-\alpha)(\lambda\dot{\gamma})^2 + \dots$$

Next we get f to the same order (f is defined in Eq. (D) of Table 8.5-1)

$$f = \frac{1 - [1 - 2\alpha(1-\alpha)(\lambda\dot{\gamma})^2 + \dots]}{1 + (1-2\alpha)[1 - 2\alpha(1-\alpha)(\lambda\dot{\gamma})^2 + \dots]} = \alpha(\lambda\dot{\gamma})^2 + \dots$$

Then the viscosity expression (of Eq. (A) in Table 8.5-1) becomes

$$\frac{\eta}{\eta_0} = \frac{(1 - \alpha(\lambda\dot{\gamma})^2 + \dots)^2}{1 + (1-2\alpha)(\alpha(\lambda\dot{\gamma})^2 + \dots)} = 1 - 2\alpha(\lambda\dot{\gamma})^2 + \dots \xrightarrow{(\lambda\dot{\gamma})^2 \rightarrow 0} 1$$

Similarly, the first normal stress coefficient (Eq. (B) of Table 8.5-1) becomes

$$\frac{\Psi_1}{2\eta_0\lambda} = \frac{(\alpha(\lambda\dot{\gamma})^2 + \dots)[1 - (\alpha(\lambda\dot{\gamma})^2 + \dots)]}{\alpha[1 - (\alpha(\lambda\dot{\gamma})^2 + \dots)]} \frac{1}{(\lambda\dot{\gamma})^2} \xrightarrow{(\lambda\dot{\gamma})^2 \rightarrow 0} 1$$

and the second normal stress coefficient (Eq. (C) of Table 8.5-1) is $\Psi_2 = -\alpha\eta_0\lambda = -\frac{1}{2}\alpha\Psi_1$ which shows (correctly) that the second normal stress coefficient is smaller than the first normal stress coefficient and has the opposite sign.

b. We begin by dividing numerator and denominator of the expression for χ^2 by $(\lambda\dot{\gamma})^2$ and then replacing $1/\lambda\dot{\gamma}$ by ε (a small quantity):

$$\chi^2 = \frac{\varepsilon\sqrt{\varepsilon^2 + 16\alpha(1-\alpha)} - \varepsilon^2}{8\alpha(1-\alpha)} = \frac{\varepsilon\sqrt{16\alpha(1-\alpha)}\sqrt{1 + [1/16\alpha(1-\alpha)]\varepsilon^2} - \varepsilon^2}{8\alpha(1-\alpha)}$$

We now expand this last expression in powers of ε :

$$\chi^2 = \frac{\varepsilon(1 + [1/32\alpha(1-\alpha)]\varepsilon^2 + \dots) - \varepsilon^2}{2\sqrt{\alpha(1-\alpha)}} = \frac{\varepsilon(1 - \varepsilon + \dots)}{2\sqrt{\alpha(1-\alpha)}}$$

Then we take the square root and get

$$\chi = \frac{\sqrt{\varepsilon(1 - \frac{1}{2}\varepsilon + \dots)}}{\sqrt{2}[\alpha(1-\alpha)]^{1/4}}$$

Then f is given by

$$f = \frac{1-\chi}{1+(1-2\alpha)\chi} = (1-\chi)[1 - (1-2\alpha)\chi + \dots] = 1 - 2\chi(1-\alpha) + \dots$$

Then the viscosity and normal-stress coefficients are

$$\begin{aligned} \frac{\eta}{\eta_0} &= \frac{(1-f)^2}{1+(1-2\alpha)f} = \frac{(2\chi(1-\alpha) + \dots)^2}{2(1-\alpha) + \dots} \\ &= \frac{(2(1-\alpha))^2 \varepsilon}{[2(1-\alpha)] \cdot 2\sqrt{\alpha(1-\alpha)}} + \dots = \sqrt{\frac{1-\alpha}{\alpha}} \frac{1}{\lambda\dot{\gamma}} + \dots \end{aligned}$$

$$\begin{aligned} \frac{\Psi_1}{2\eta_0\lambda} &= \frac{f(1-\alpha f)}{\alpha(1-f)} \left(\frac{1}{\lambda\dot{\gamma}}\right)^2 = \frac{(1-\alpha) + \dots}{\alpha[2\chi(1-\alpha) + \dots]} \left(\frac{1}{\lambda\dot{\gamma}}\right)^2 \\ &= \frac{[\alpha(1-\alpha)]^{1/4}}{\sqrt{2}\alpha} \left(\frac{1}{\lambda\dot{\gamma}}\right)^{3/2} + \dots \end{aligned}$$

$$\frac{\Psi_2}{\eta_0 \lambda} = -\left(\frac{1}{\lambda \dot{\gamma}}\right)^2 + \dots$$

These expressions show (correctly) that the normal stress coefficient has a steeper slope at high shear rates than the viscosity. They also give a second normal stress coefficient that is smaller than the first normal stress coefficient, and that the two coefficients have opposite signs (this is in agreement with the experimental data for flexible polymers).

c. For elongational flow, we first get the limit as $\lambda \dot{\epsilon} \rightarrow 0$ (please note that this $\dot{\epsilon}$ is not related to the ϵ in part (b)). In this limit we can expand the square root signs that appear in Eq. (H) of Table 8.5-1 thus:

$$\begin{aligned} \frac{\bar{\eta}}{3\eta_0} &= \frac{1}{6\alpha} \left[3 + \frac{1}{\lambda \dot{\epsilon}} (1 - 2(1 - 2\alpha)\lambda \dot{\epsilon} + \dots - 1 - (1 - 2\alpha)\lambda \dot{\epsilon} - \dots) \right] \\ &= \frac{1}{6\alpha} \left[3 + \frac{1}{\lambda \dot{\epsilon}} (-3(1 - 2\alpha)\lambda \dot{\epsilon} + \dots) \right] = 1 + \dots \end{aligned}$$

In the limit that $\lambda \dot{\epsilon}$ becomes infinite, the quadratic terms under the square root signs dominate and we get

$$\frac{\bar{\eta}}{3\eta_0} = \frac{1}{6\alpha} (3 + 2 - 1) = \frac{2}{3\alpha}$$

Thus, the elongational viscosity remains finite, unlike the Oldroyd model for which the elongational viscosity becomes infinite.

8C.1 The cone-and-plate viscometer

a. According to Eq. 2B.11-1, the velocity distribution in the cone-and-plate system is given approximately by

$$\frac{v_\phi}{r} = \Omega \left(\frac{(\pi/2) - \theta}{\psi_0} \right)$$

The shear rate is given by the relation two lines above Eq. 8.3-2. In this problem the only nonzero components of the rate of strain tensor are $\dot{\gamma}_{\theta\phi}$ and $\dot{\gamma}_{\phi\theta}$. This can be seen in one of two ways: (a) Look at the right side of Eqs. B.1-15 to 21, but without the factor $-\mu$ and with the $\text{div } \mathbf{v}$ terms set equal to zero; these are the spherical components of the rate of strain tensor given in Eq. 8.3-1 or Eq. 8.4-1; or (b) add to the components of $\text{grad } \mathbf{v}$ in Eqs. (S) to (AA) in Table A.7-3 the corresponding transposes in order to construct the rate-of-strain tensor according to Eq. 8.3-1. Therefore, from Eq. B.1-19

$$\dot{\gamma}_{\theta\phi} = \dot{\gamma}_{\phi\theta} = \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) \approx \frac{1}{r} \frac{\partial v_\phi}{\partial \theta}$$

since the angle between the cone and plate is extremely small, which means that $\sin \theta \approx \sin \frac{1}{2} \pi = 1$. When the velocity distribution of Eq. 2B.11-1 is inserted in to the above approximate expression for the nonzero components of the rate-of-strain tensor we get

$$\dot{\gamma}_{\theta\phi} = \dot{\gamma}_{\phi\theta} = \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{r} \right) = -\frac{\Omega}{\psi_0}$$

Hence the shear rate is

$$\dot{\gamma} = \sqrt{\frac{1}{2} (\dot{\gamma}_{\theta\phi} \dot{\gamma}_{\phi\theta} + \dot{\gamma}_{\phi\theta} \dot{\gamma}_{\theta\phi})} = \pm \dot{\gamma}_{\theta\phi} = \mp \frac{\Omega}{\psi_0}$$

We now have to choose the proper sign. Both Ω and ψ_0 are positive quantities. Therefore the plus sign must be selected.

b. The non-Newtonian viscosity is obtained from the ratio $\eta = (\tau_{\theta\phi} / -\dot{\gamma}_{\theta\phi})$. Hence we have to find a way to get the shear stress

from the measured quantities. As pointed out in (a), since the shear rate is constant throughout the gap, all stress components are also constant. This means that the torque can be calculated from the shear stress by integrating shear-stress times lever arm over the surface of the plate from zero to R :

$$T_z = \int_0^{2\pi} \int_0^R r \cdot \tau_{\theta\phi} \Big|_{\theta=\pi/2} r dr d\phi = \frac{2}{3} \pi R^3 \tau_{\theta\phi} \quad \text{whence} \quad \tau_{\theta\phi} = \frac{3T_z}{2\pi R^3}$$

Then the non-Newtonian viscosity is given by

$$\eta = \frac{\tau_{\theta\phi}}{-\dot{\gamma}_{\theta\phi}} = \frac{(3T_z/2\pi R^3)}{(\Omega/\psi_0)} = \frac{3T_z \psi_0}{2\pi R^3 \Omega}$$

c. Equation 8C.1-2 follows directly from Eq. B.5-7, after dropping the terms on the left side and the $\tau_{\theta r}$ and $\tau_{\phi r}$ terms on the right side; the terms $\tau_{\theta\theta}$ and $\tau_{\phi\phi}$ must, however, be retained, since we know that normal stresses are nonzero in shear flow. Then Eq. 8C.1-3 follows from $\pi_{rr} = p + \tau_{rr}$ and some minor rearranging, and Eq. 8C.1-4 follows from the definitions of the normal stress coefficients and from the results of (a).

Next we integrate Eq. 8C.1-4 from the outer rim of the cone-plate system to some arbitrary position r :

$$\int_R^r d\pi_{\theta\theta} = -[(\Psi_1 + 2\Psi_2)\dot{\gamma}^2] \int_R^r d \ln r$$

The normal-stress coefficients are constants, because of the constancy of the shear rate over the gap. Therefore we get

$$\begin{aligned} \pi_{\theta\theta}(r) &= \pi_{\theta\theta}(R) - [(\Psi_1 + 2\Psi_2)\dot{\gamma}^2] \ln \frac{r}{R} \\ &= \pi_{rr}(R) + [\pi_{\theta\theta}(R) - \pi_{rr}(R)] - [(\Psi_1 + 2\Psi_2)\dot{\gamma}^2] \ln \frac{r}{R} \\ &= p_a - \Psi_2 \dot{\gamma}^2 - [(\Psi_1 + 2\Psi_2)\dot{\gamma}^2] \ln \frac{r}{R} \end{aligned}$$

Here we have used the boundary condition that the normal stress at the rim is equal to the atmospheric pressure, and we have also used the definition of the second normal stress difference.

d. The force exerted by the fluid in the z -direction on the cone is then obtained from the result in (c) as follows:

$$\begin{aligned}
 F_z &= \int_0^{2\pi} \int_0^R \pi_{\theta\theta}(r) r dr d\phi - \pi R^2 p_a \\
 &= 2\pi \int_0^R \left[p_a - \Psi_2 \dot{\gamma}^2 - [(\Psi_1 + 2\Psi_2) \dot{\gamma}^2] \ln \frac{r}{R} \right] r dr - \pi R^2 p_a \\
 &= -\pi R^2 \Psi_2 \dot{\gamma}^2 - 2\pi R^2 [(\Psi_1 + 2\Psi_2) \dot{\gamma}^2] \int_0^1 (\ln \xi) \xi d\xi \\
 &= -\pi R^2 \Psi_2 \dot{\gamma}^2 - 2\pi R^2 [(\Psi_1 + 2\Psi_2) \dot{\gamma}^2] \left(\frac{1}{2} \xi^2 \ln \xi - \frac{1}{4} \xi^2 \right) \Big|_0^1 \\
 &= -\pi R^2 \Psi_2 \dot{\gamma}^2 + \frac{1}{2} \pi R^2 \Psi_1 \dot{\gamma}^2 + \pi R^2 \Psi_2 \dot{\gamma}^2 = \frac{1}{2} \pi R^2 \Psi_1 \dot{\gamma}^2
 \end{aligned}$$

Here we made use of the fact that $\lim_{\xi \rightarrow 0} \xi \ln \xi = 0$. Solving for the first normal stress difference, we get finally

$$\Psi_1(\dot{\gamma}) = \frac{2F_z}{\pi R^2 \dot{\gamma}^2}$$

e. If one measures $\pi_{\theta\theta}(r) - p_a$ as a function of r/R using flush-mounted pressure transducers, then, knowing Ψ_1 from (d), the second normal-stress coefficient can be calculated from Eq. 8C.1-5.

8C.2 Squeezing flow between parallel disks

This problem is solved by a quasi-steady-state method. Conservation of mass states that, for an incompressible fluid, the mass rate at which the fluid crosses the cylindrical surface at r should equal the rate at which the mass between the plates within the cylindrical surface at r decreases. The rate of mass displacement caused by the disk motion is:

$$w(r) = \pi r^2 \rho (-\dot{H}) \quad \text{where} \quad (\dot{H} = dh/dt)$$

To get the mass rate of flow between the two disks, we adapt the result in Eq. 8.3-14 by making the following correspondences locally: $W \rightarrow 2\pi r$, $2B \rightarrow H$, $(\mathcal{P}_0 - \mathcal{P}_L)/L \rightarrow -dp/dr$, and $w \rightarrow w(r)$. Then we get for the mass rate of flow emerging from between the disks:

$$w(r) = \frac{2(2\pi r)\left(\frac{1}{2}H\right)^2}{(1/n)+2} \left(-\frac{dp}{dr} \frac{(\frac{1}{2}H)}{m} \right)^{1/n}$$

Equating the two expressions for $w(r)$ we get a differential equation for for the pressure distribution $p(r)$ between the disks

$$-\frac{dp}{dr} = \frac{[r(-\dot{H})]^n [(1/n)+2]^n}{[H^2]^n} \left(\frac{2m}{H} \right)$$

This may be integrated to give

$$-\int_p^{p_{\text{atm}}} dp = \frac{2m(-\dot{H})^n [(1/n)+2]^n}{H^{2n+1}} \int_r^R \bar{r}^n d\bar{r}$$

or

$$p - p_{\text{atm}} = \frac{2m(-\dot{H})^n [(1/n)+2]^n}{H^{2n+1}} \frac{R^{n+1}}{n+1} \left[1 - \left(\frac{r}{R} \right)^{n+1} \right]$$

which is the power-law equivalent of the Newtonian result in Eq. 3C.1-13.

When a constant force F_0 is applied to the upper disk, this force must be resisted by the pressure in the fluid at the upper disk, integrated over the disk (we include here also the normal stress τ_{zz} , even though we know that it will not contribute for a generalized Newtonian fluid--the proof for this is similar to that given in Example 3.1-1 for Newtonian fluids):

$$\begin{aligned} F_0 &= \int_0^{2\pi} \int_0^R (p - p_0 + \tau_{zz}) r dr d\theta \\ &= 2\pi \cdot \frac{2m(-\dot{H})^n \left[\left(\frac{1}{n} \right) + 2 \right]^n R^{n+3}}{H^{2n+1}} \int_0^1 (1 - \xi^2) \xi d\xi \\ &= \frac{2\pi m (-\dot{H})^n \left[\left(\frac{1}{n} \right) + 2 \right]^n R^{n+3}}{H^{2n+1}} \frac{1}{n+3} \end{aligned}$$

This is now a differential equation for the motion of the upper plate as a function of time:

$$-\frac{\dot{H}}{H^{(1/n)+2}} = \left(\frac{(n+3)}{2\pi m R^{n+3}} \right)^{1/n} \left(\frac{1}{\left(\frac{1}{n} \right) + 2} \right) F_0^{1/n}$$

This can be integrated to give:

$$-\int_{H_0}^H \frac{dH}{H^{(1/n)+2}} = \left(\frac{(n+3)}{2\pi m R^{n+3}} \right)^{1/n} \left(\frac{1}{\left(\frac{1}{n} \right) + 2} \right) F_0^{1/n} \int_0^t dt$$

or

$$\frac{1}{H^{(1/n)+1}} - \frac{1}{H_0^{(1/n)+1}} = \left(\frac{(n+3)}{2\pi m R^{n+3}} \right)^{1/n} \left(\frac{\left(\frac{1}{n} \right) + 1}{\left(\frac{1}{n} \right) + 2} \right) F_0^{1/n} t$$

This simplifies properly to Eq. 3C.1-16 for the Newtonian fluid.

8C.3 Verification of Giesekus viscosity function

a. Eq. 8.5-4 is the same as Eq. 8.5-3 with $\lambda_1 = \mu_1 = \lambda$ and $\lambda_2 = \mu_2 = \mu_0 = 0$, but contains in addition a term $-(\lambda/\eta_0)\alpha\{\boldsymbol{\tau} \cdot \boldsymbol{\tau}\}$. Therefore, for steady shear flow, we may take over Eqs. 8.5-5 to 8 by making the appropriate changes in the constants, and by adding the extra term. To do the latter we need to calculate the components of the product $\{\boldsymbol{\tau} \cdot \boldsymbol{\tau}\}$; this is most easily done by matrix multiplication:

$$\begin{pmatrix} \tau_{xx} & \tau_{yx} & 0 \\ \tau_{yx} & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{pmatrix} \begin{pmatrix} \tau_{xx} & \tau_{yx} & 0 \\ \tau_{yx} & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{pmatrix} = \begin{pmatrix} \tau_{xx}^2 + \tau_{yx}^2 & \tau_{yx}(\tau_{xx} + \tau_{yy}) & 0 \\ \tau_{yx}(\tau_{xx} + \tau_{yy}) & \tau_{xx}^2 + \tau_{yy}^2 & 0 \\ 0 & 0 & \tau_{zz}^2 \end{pmatrix}$$

When we use this result, Eqs. 8.5-5 to 8 become modified as follows for the Giesekus model:

$$\begin{aligned} \tau_{xx} - 2\lambda\dot{\gamma}\tau_{yx} - (\lambda/\eta_0)\alpha(\tau_{xx}^2 + \tau_{yx}^2) &= 0 \\ \tau_{yy} - (\lambda/\eta_0)\alpha\tau_{xx}^2 + \tau_{yx}^2 &= 0 \\ \tau_{zz} - (\lambda/\eta_0)\alpha\tau_{zz}^2 &= 0 \\ \tau_{yx} - \lambda\dot{\gamma}\tau_{yy} - (\lambda/\eta_0)\alpha\tau_{yx}(\tau_{xx} + \tau_{yy}) &= -\eta_0\dot{\gamma} \end{aligned}$$

When these equations are multiplied by λ/η_0 and dimensionless variables are introduced, we obtain Eqs. 8C.3-1 plus an equation that gives $T_{zz} = 0$ (this equation gives another solutions which is physically unacceptable).

b. When these dimensionless equations are written in terms of the dimensionless normal stresses we get

$$\begin{aligned} N_1[1 - \alpha(N_1 + 2N_2)] &= 2T_{yx}\dot{\Gamma} \\ N_2 &= \alpha(T_{yx}^2 + N_2^2) \\ T_{yx}[1 - \alpha(N_1 + 2N_2)] &= -(1 - N_2)\dot{\Gamma} \end{aligned}$$

c. The second equation in (b) can be solved at once for the dimensionless shear stress:

$$T_{yx}^2 = \frac{N_2}{\alpha}(1 - \alpha N_2) \quad (*)$$

Division of the first of the equations in (b) by the third gives a relation for the dimensionless first normal stress difference in terms of the dimensionless second normal stress difference and the dimensionless shear stress; into that we can substitute the equation just obtained above and get

$$N_1 = -\frac{2N_2(1 - \alpha N_2)}{\alpha(1 - N_2)} \quad (**)$$

Next, we square the third equation in (b), and then insert the expression for T_{yx}^2 from (*) and the expression for N_1 from (**), to get

$$\dot{\Gamma}^2 = \frac{N_2(1 - \alpha N_2)[1 + (1 - 2\alpha)N_2]^2}{\alpha(1 - N_2)^4} \quad (***)$$

d. To solve the final equation in (c) Giesekus (see p 87 of Ref. 5 on p. 262) suggested making the following change of variable:

$$N_2 = \frac{1 - \chi}{1 + (1 - 2\alpha)\chi}$$

Then the various factors in the final equation in (c) are:

$$1 - N_2 = \frac{2\chi(1 - \alpha)}{1 + (1 - 2\alpha)\chi}$$

$$1 - \alpha N_2 = \frac{(1 + \chi)(1 - \alpha)}{1 + (1 - 2\alpha)\chi};$$

$$1 + (1 - 2\alpha)N_2 = \frac{2(1 - \alpha)}{1 + (1 - 2\alpha)\chi}$$

When the last four equations are substituted into (***) we get

$$\dot{\Gamma}^2 = \frac{1 - \chi^2}{4\alpha(1 - \alpha)\chi^4}$$

This is a quadratic equation for χ^2 which may be readily solved:

$$\chi^2 = \frac{\sqrt{1 + 16\alpha(1 - \alpha)\dot{\Gamma}^2} - 1}{8\alpha(1 - \alpha)\dot{\Gamma}^2} = 1 - 4\alpha(1 - \alpha)\dot{\Gamma}^2 + \dots$$

Then this result along with Eq. 8C.3-7 gives the second normal stress difference as a function of the shear rate. Having this, the shear stress and first normal stress difference can be obtained from Eqs. 8C.3-4 and 5.

As the shear rate goes to zero, one gets $\eta \rightarrow \eta_0$, $\Psi_1 \rightarrow 2\eta_0\lambda$, and $\Psi_2 \rightarrow -\alpha\eta_0\lambda$. The functions given in Table 8.5-1 are given in Giesekus's paper as well as in R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Vol. 1 (1987). pp. 361-368.

8C.4 Tube flow for the Oldroyd 6-constant model

The non-Newtonian viscosity for the Oldroyd model is given in Eq. 8.5-9. Therefore for tube flow

$$\tau_{rz} = -\eta \dot{\gamma}_{rz} = \eta_0 \dot{\gamma} \left(\frac{1 + \sigma_2 \dot{\gamma}^2}{1 + \sigma_1 \dot{\gamma}^2} \right)$$

From Eq. 2.3-13, we have the expression valid for any kind of liquid

$$\tau_{rz} = \frac{(\mathfrak{P}_0 - \mathfrak{P}_L)r}{2L}$$

Combination of these two results gives

$$r = \frac{2\eta_0 \dot{\gamma} L}{\mathfrak{P}_0 - \mathfrak{P}_L} \left(\frac{1 + \sigma_2 \dot{\gamma}^2}{1 + \sigma_1 \dot{\gamma}^2} \right)$$

The mass rate of flow through the tube is given by

$$w = \rho \int_0^{2\pi} \int_0^R v_z r dr d\theta = 2\pi\rho \int_0^R v_z r dr = 2\pi\rho \left[\frac{1}{2} v_z r^2 \Big|_0^R - \int_0^R \frac{1}{2} r^2 \frac{dv_z}{dr} dr \right]$$

The first term in the last result is zero at both limits. We next integrate by parts again

$$w = -\pi\rho \left[\frac{1}{3} r^3 \frac{dv_z}{dr} \Big|_0^R - \int_0^R \frac{1}{3} r^3 \frac{d}{dr} \left(\frac{dv_z}{dr} \right) dr \right] = \frac{1}{3} \pi\rho R^3 \dot{\gamma}_R - \frac{1}{3} \pi\rho \int_0^{\dot{\gamma}_R} [r(\dot{\gamma})]^3 d\dot{\gamma}$$

in which $\dot{\gamma}_R$ is $-dv_z/dr$ evaluated at the tube wall. This result is good for any non-Newtonian fluid. We now specialize to the Oldroyd model:

$$\begin{aligned} w &= \frac{1}{3} \pi\rho R^3 \dot{\gamma}_R - \frac{1}{3} \pi\rho \left(\frac{2\eta_0 L}{\mathfrak{P}_0 - \mathfrak{P}_L} \right)^3 \int_0^{\dot{\gamma}_R} \left[\frac{1 + \sigma_2 \dot{\gamma}^2}{1 + \sigma_1 \dot{\gamma}^2} \right]^3 \dot{\gamma}^3 d\dot{\gamma} \\ &= \frac{1}{3} \pi\rho R^3 \dot{\gamma}_R - \frac{1}{3} \pi\rho \left(\frac{2\eta_0 L}{\mathfrak{P}_0 - \mathfrak{P}_L} \right)^3 \int_0^{\sigma_1 \dot{\gamma}_R^2} \left(\frac{1 + nY}{1 + Y} \right)^3 Y dY \frac{1}{2\sigma_1^2} \end{aligned}$$

in which $Y = \sigma_1 \dot{\gamma}^2$ and $n = \sigma_2 / \sigma_1$.

The expression for r given earlier may be evaluated at the tube wall, and this gives

$$R = \frac{2\eta_0 \dot{\gamma}_R L}{\mathfrak{P}_0 - \mathfrak{P}_L} \left(\frac{1 + \sigma_2 \dot{\gamma}_R^2}{1 + \sigma_1 \dot{\gamma}_R^2} \right) = \left(\frac{2\eta_0 L}{\mathfrak{P}_0 - \mathfrak{P}_L} \right) \sqrt{X} \left(\frac{1 + nX}{1 + X} \right) \frac{1}{\sqrt{\sigma_1}} \quad (*)$$

in which $X = \sigma_1 \dot{\gamma}_R^2$. Next, multiply the equation for w by $3\sqrt{\sigma_1} / \pi \rho R^3$ and eliminate $\mathfrak{P}_0 - \mathfrak{P}_L$ by using the expression for R just above to get

$$\Omega = \frac{3\sqrt{\sigma_1} w}{\pi \rho R^3} = \sqrt{X} - \frac{1}{2X^{3/2}} \left(\frac{1 + X}{1 + nX} \right)^3 f(X) \quad (**)$$

in which

$$\begin{aligned} f(X) &= \int_0^X \left(\frac{1 + nY}{1 + Y} \right)^3 Y dY \\ &= \frac{1}{2} n^3 X^2 - 3n^2 (n-1) X + 3n(n-1)(2n-1) \ln(1+X) \\ &\quad - \frac{1}{2} X \left(\frac{n-1}{1+X} \right)^2 [6n + (7n-1)X] \end{aligned} \quad (***)$$

This gives, in dimensionless form, the mass rate of flow in terms of the wall shear rate. The latter may be eliminated in favor of the pressure drop by using the expression for R above. That is, to get Ω in terms of the pressure drop, the asterisked equation have to be combined. The curves thus obtained may be found in the original publications (refs. 6 and 7).

8C.5 Chain models with rigid connectors

a. The paper by M. Gottlieb is the first example of a molecular dynamics calculation for polymer chain with rigid connectors. He followed the motions of a three-bead-two-rod model as it moves around in a liquid made up of 47 "solvent beads." The solution is presumed to be at macroscopic equilibrium. He showed that one does not get a Gaussian distribution of orientations for the polymer model, in agreement with the previously published theory of H. A.

Kramers, Physica 11, 1-10 (1944). See also D. P. Bird, C. F. Curtiss,

R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids* (Vol. 2), Wiley, New York, 2nd edition (1987), pp. 40-41. This book will hereinafter be referred to as "DPL."

b. The publication by O. Hassager showed how to get the viscosity and normal stress coefficients for a three-bead-two-rod chain in a flow situation. In a subsequent publication [C. F. Curtiss and R. B. Bird, *J. Non-Newtonian Fluid Mechanics*, 2, 392-396 (1977)] it was shown how to extend the calculations to models with 4 and 5 beads. Hassager showed that the viscosity and first normal stress coefficient for models with infinitely stiffened Fraenkel springs are different from those of the rigid rod models. This perplexing situation was what led Gottlieb to investigate the same problem by using molecular dynamics. See also DPL (§16.5).

c. The paper by X. J. Fan and T. W. Liu dealt with Kirkwood-Riseman chains of 3 to 8 beads. This model has fixed bond lengths and fixed bond angles, that is, two kinds of constraints. They were able to evaluate the equilibrium configurational distribution, and they found that it differs from the classical "Gaussian distribution." This calculation enabled the authors to relate rheological properties to "chain stiffness," by comparing their results with those for a Kramers freely jointed chain. See also DPL (§16.6).

d. The paper by T. W. Liu is a landmark contribution to the

polymer molecules "unravel" under the influence of the elongational flow.

e. The paper by H. H. Saab, et al., is an extensive comparison between the Curtiss-Bird phase-space theory for polymer melts and the available experimental data on a host of rheological properties. An extensive comparison of the results with those of the theory of Doi and Edwards confirms that in almost every instance, the Curtiss-Bird theory is to be preferred over the Doi-Edwards theory. In the Curtiss-Bird theory, the polymer molecules are modeled as Kramers freely jointed bead-rod chains, whereas in the Doi-Edwards theory, mixed modeling is used. See also DPL Chapter 19.

f. The papers by J. D. Schieber show how the extension of the Curtiss-Bird theory to polydisperse polymer melts can be implemented. Both the log-normal (Wesslau) distribution of molecular weights and the Flory-Schultz distribution are used. The curves for the viscosity, first normal-stress coefficient, elongational stress growth viscosity, and steady-state elongational viscosity were obtained. These curves often differed appreciably from those for monodisperse samples. Comparisons with experimental data are given in the second of the two papers. See DPL, Example 19.6-1.

General Comments:

i. The rodlike connectors are generally much more difficult to deal with than the springlike connectors, because it becomes necessary to use nonorthogonal coordinates to describe the chain space. For Hookean springs, one can perform many of the kinetic theory derivations with relative ease. For non-Hookean springs, it is possible to make some assumptions that enable analytical results to be obtained. But even for rigid dumbbells and three-bead-two-rod models, analytical calculations become prohibitively time consuming.

ii. To overcome the problems associated with the use of rigid connectors in modeling, molecular dynamics and Brownian dynamics have proven to be useful. Also, Brownian dynamics can be useful when considering the flow in constrained channels, where the interaction with the containing walls have to be considered. Brownian dynamics proves to be particularly helpful in getting information about the actual motions of the polymer molecules during various types of flow.

9A.1 Prediction of thermal conductivities of gases at low density.

a. Since Argon is monatomic, we use Eq. 9.3-13 to predict its k in the low-density gas region:

$$k = 1.9891 \times 10^{-4} \frac{\sqrt{T/M}}{\sigma^2 \Omega_\mu}$$

Here $T = 100 + 273.15 = 373.15\text{K}$, and Table E.1 gives $M = 39.948$, $\sigma = 3.432 \text{ \AA}$, $\epsilon/\text{K} = 122.4\text{K}$ for Argon. Then $\kappa T/\epsilon = 373.15/122.4 = 3.049$, and interpolation in Table E.2 gives $\Omega_\mu = 1.0344$. Equation 9.3-13 then gives

$$k = 1.9891 \times 10^{-4} \frac{\sqrt{373.15/39.948}}{3.432^2 \times 1.0344} = 499 \times 10^{-7} \text{ cal/s}\cdot\text{cm}\cdot\text{K}$$

which is within 1.5 percent of the observed value.

b. Equation 9.3-15, Eucken's formula, gives

$$k = \left(\hat{C}_p + 1.25R/M \right) \mu = (\tilde{C}_p + 1.25R)\mu/M$$

with $R = 1.987 \text{ cal/g}\cdot\text{mole}\cdot\text{K}$. Insertion of the data for \tilde{C}_p and μ , and for M from Table E.1, gives

$$\begin{aligned} \text{for NO, } k &= (7.15 + 1.25 * 1.987) \frac{1929 \times 10^{-7}}{30.01} = 620 \times 10^{-7} \text{ cal/s}\cdot\text{cm}\cdot\text{K} \\ &= 0.02595 \text{ W/m}\cdot\text{K} \\ &\text{vs. } 0.02590 \text{ W/m}\cdot\text{K} \text{ from Table 9.1-2.} \end{aligned}$$

$$\begin{aligned} \text{for CH}_4, k &= (8.55 + 1.25 \times 1.987) \frac{1116 \times 10^{-7}}{16.04} = 768 \times 10^{-7} \text{ cal/s}\cdot\text{cm}\cdot\text{K} \\ &= 0.03212 \text{ W/m}\cdot\text{K} \\ &\text{vs. } 0.03427 \text{ W/m}\cdot\text{K} \text{ from Table 9.1-2.} \end{aligned}$$

9A.2 Computation of the Prandtl number for gases at low density.

Use of Eq. 9.3-16, with molar heat capacities $\tilde{C}_p = M\hat{C}_p$ calculated from the given values \hat{C}_p and molecular weights M from Table E.2, along with $R = 8.31451 \times 10^3$ J/kg-mol·K from Appendix F, gives the predictions in column (a) of the following table. Computation of Pr from its definition, Eq. 9.1-9, and the tabulated \hat{C}_p , μ , and k , gives the values in column (b). The predictions are closely confirmed for He and Ar, but are less successful for the polyatomic compounds.

Gas	(a) Pr from Eq. 9.3-16	(b) Pr from \hat{C}_p , μ , and k
He	0.667	0.670
Ar	0.667	0.665
H ₂	0.735	0.714
Air	0.736	0.710
CO ₂	0.782	0.769
H ₂ O ($M = 18.016$)	0.764	0.862

9A.3 Estimation of the thermal conductivity of a dense gas.

a. Table E.1 gives the following critical constants for methane (CH_4): $T_c = 191.1$ K, $p_c = 45.8$ atm, and $k_c = 158 \times 10^{-6}$ cal/cm·s·K. The reduced conditions for the prediction are then $T_r = (459.7 + 127)/(1.8 \times 191.1) = 1.71$ and $p_r = 110.4/45.8 = 2.41$. From Fig. 9.2-1 we find $k_r = 0.77$ at that state, giving

$$\begin{aligned}k &= k_r k_c = 0.77 \times 158 \times 10^{-6} \\ &= 1.22 \times 10^{-4} \text{ cal/cm}\cdot\text{s}\cdot\text{K} \\ &= 0.0294 \text{ Btu/hr}\cdot\text{ft}\cdot\text{F}.\end{aligned}$$

which is about 4% above the observed value.

b. For this calculation, we need to predict the viscosity of methane at 127 F (325.9 K) and low pressure from Eq. 1.4-18. We find $\sigma = 3.780$ and $\kappa T/\varepsilon = 325.9/154 = 2.116$ and $\Omega_\mu = 1.153$ by use of Tables E.1 and E.2. Hence,

$$\mu = 2.6693 \times 10^{-5} \frac{\sqrt{16.04 \times 325.9}}{(3.780)^2 \times 1.153} = 1171 \times 10^{-7} \text{ g/cm}\cdot\text{s}$$

Next we use the Eucken formula, Eq. 9.3-15, to estimate the thermal conductivity k° at low pressure and 127 F, where $\tilde{C}_p = 37.119$ J/g·mol·K according to the heat capacity polynomial given for methane in Reid, Prausnitz and Poling (1987):

$$\begin{aligned}k^\circ &= (37.119 + 1.25 \times 8.31451) \frac{1171 \times 10^{-7}}{16.04} = 0.000347 \text{ W/cm}\cdot\text{K} \\ &= 0.0347 \text{ W/m}\cdot\text{K} \\ &= 0.0200 \text{ Btu/hr}\cdot\text{ft}\cdot\text{F}\end{aligned}$$

Finally, we multiply k° by the ratio of k_r at 110.4 atm to the asymptote $k_r = 0.52$ at $p_r = 0$ in Fig. 9.2-1. The resulting predicted k at (110.4 atm, 127 F) is

$$k = 0.0200 \times 0.77/0.52 = 0.0297 \text{ Btu/hr}\cdot\text{ft}\cdot\text{F}$$

and is just 1% above the measured value. This is unusually good agreement.

9A.4 Prediction of the thermal conductivity of a gas mixture.

The data for this problem are as follows:

Component	M	$\mu \times 10^5$, Pa·s	k , W/m·K	Mole fraction
1(H ₂)	2.016	0.8944	0.1789	0.80
2(CO ₂)	44.01	1.506	0.01661	0.20

Insertion of these data into Eq. 1.4-16 gives the dimensionless coefficients

$$\Phi_{11} = \Phi_{22} = 1.0$$

$$\begin{aligned}\Phi_{12} &= \frac{1}{\sqrt{8}} \left(1 + \frac{2.016}{44.01} \right)^{-1/2} \left[1 + \left(\frac{0.8944}{1.506} \right)^{1/2} \left(\frac{44.01}{2.016} \right)^{1/4} \right]^2 \\ &= 2.457\end{aligned}$$

$$\begin{aligned}\Phi_{21} &= \frac{1}{\sqrt{8}} \left(1 + \frac{44.01}{2.016} \right)^{-1/2} \left[1 + \left(\frac{1.506}{0.8944} \right)^{1/2} \left(\frac{2.016}{44.01} \right)^{1/4} \right]^2 \\ &= 0.1819\end{aligned}$$

Substitution of these results into Eq. 9.3-17 gives

$$\begin{aligned}k_{\text{mix}} &= \frac{0.8 \times 0.1789}{0.8 \times 1.0 + 0.2 \times 2.457} + \frac{0.2 \times 0.01661}{0.8 \times 0.1819 + 0.2 \times 1.0} \\ &= 0.1204 \text{ W/m}\cdot\text{K}\end{aligned}$$

9A.5 Estimation of the thermal conductivity of a pure liquid.

We first calculate the derivative $(\partial p/\partial \rho)_T$ required for Eq. 9.3-4:

$$\begin{aligned}(\partial p/\partial \rho)_T &= \rho^{-1} / [\rho^{-1}(\partial \rho/\partial p)_T]^{-1} \\ &= (1/0.9938)[38 \times 10^{-6}]^{-1} = 2.648 \times 10^4 \text{ megabar cm}^3/\text{g} \\ &= 2.648 \times 10^{10} \text{ cm}^2/\text{s}^2\end{aligned}$$

Inserting this result into Eq. 9.3-4 and setting $C_p \approx C_v$, we obtain

$$v_s = \sqrt{2.648 \times 10^{10}} = 1.627 \times 10^5 \text{ cm/s}$$

Equation 9.4-3 then gives the following estimate of the thermal conductivity:

$$\begin{aligned}k &= 2.80(\tilde{N}\rho/M)^{2/3} \kappa v_s \\ &= 2.80 \left[\frac{6.02214 \times 10^{23} \times 0.9938}{18.02} \right]^{2/3} \times 1.38066 \times 10^{-16} \times 1.627 \times 10^5 \\ &= 6.50 \times 10^4 \text{ g cm/s}^3 \cdot \text{K} \\ &= 0.650 \text{ W/m} \cdot \text{K} \\ &= 0.375 \text{ Btu/hr} \cdot \text{ft} \cdot \text{F}\end{aligned}$$

9A.6 Calculation of the Lorenz number.

a. When κ and e in Eq. 9A.6-1 are expressed in terms of the gas constant R and Faraday constant F , the Lorenz number takes the form

$$L = \frac{\pi^2}{3} \left(\frac{R}{F} \right)^2$$

Insertion of numerical values for R and F from Appendix E gives

$$\begin{aligned} L &= \frac{\pi^2}{3} \left(\frac{8.31451}{96485.3} \right)^2 \\ &= 2.44 \times 10^{-8} \text{ volt}^2/\text{K}^2 \end{aligned}$$

b. Insertion of the result just found, and the given k_e and T , into Eq. 9.5-1 gives the thermal conductivity estimate

$$\begin{aligned} k &= Lk_eT \\ &= \frac{2.44 \times 10^{-8} \text{ volt}^2/\text{K}^2 \times 293.15\text{K}}{1.72 \times 10^{-6} \text{ ohm}\cdot\text{cm}} \\ &= 4.16 \text{ volt}^2/\text{K}\cdot\text{ohm}\cdot\text{cm} \\ &= 416 \text{ W/m}\cdot\text{K} \end{aligned}$$

for copper at 20°C.

9A.7. Corroboration of the Wiedemann-Franz-Lorenz Law.

Conversion of the tabulated data into SI units and insertion into Eq. 9.5-1 gives the following results at $T=293.15\text{K}$:

Metal	$1/k_e$, ohm·m	k , W/m·K	Lorenz number, $L = k/k_e T$, volt ² /K ²
Na	4.6×10^4	133	2.1×10^{-8}
Ni	6.9×10^4	59	1.4×10^{-8}
Cu	1.69×10^4	385	2.2×10^{-8}
Al	2.62×10^4	209	1.9×10^{-8}

The approximate agreement of L for these metals illustrates the Wiedemann-Franz-Lorenz law.

9A.8 Thermal conductivity and Prandtl number of a polyatomic gas.

a. To calculate k for a polyatomic gas at moderate pressure we use Eq. 9.3-15,

$$k = (\hat{C}_p + 1.25R/M)\mu = (\tilde{C}_p + 1.25R)\mu/M$$

along with the viscosity expression in Eq. 1.4-18:

$$\mu = 2.6693 \times 10^{-5} \frac{\sqrt{MT}}{\sigma^2 \Omega_\mu}$$

From Table E.1 we find, for CH_4 , the values $M = 16.04$, $\sigma = 3.780$, $\epsilon/\text{K} = 154\text{K}$, and from Table E.2 at $kT/\epsilon = 1500/154 = 9.740$ we find $\Omega_\mu = 0.8280$. Equation 1.4-18 then gives the predicted viscosity

$$\begin{aligned} \mu &= 2.6693 \times 10^{-5} \frac{\sqrt{16.04 \times 1500}}{(3.780)^2 \times 0.8280} \\ &= 3500 \times 10^{-7} \text{ g/cm}\cdot\text{s} \end{aligned}$$

and Eq. 9.315 gives the predicted k value

$$\begin{aligned} k &= (20.71 + 1.25 \times 1.987) \times 3500 \times 10^{-7} / 16.04 \\ &= 5.06 \times 10^{-4} \text{ cal/cm}\cdot\text{s}\cdot\text{K} \\ &= 2.12 \text{ W/m}\cdot\text{K} \end{aligned}$$

b. The predicted Prandtl number according to Eq. 9.3-16 is

$$\text{Pr} = \frac{20.71}{20.71 + 1.25 \times 1.987} = 0.89, \text{ dimensionless}$$

9A.9 Thermal conductivity of gaseous chlorine.

For Cl₂, Table E.1 gives $M = 70.91$, $\sigma = 4.115 \text{ \AA}$ and $\varepsilon/K = 357\text{K}$. Equation 1.4-14 then gives

$$\mu = 2.6693 \times 10^{-5} \frac{\sqrt{MT}}{\sigma^2 \Omega_\mu} = 1.34274 \times 10^{-5} \sqrt{T}$$

with $\mu [=] \text{g/cm}\cdot\text{s}$ and $T [=] \text{K}$, and Eq. 9.3-15 gives $k = (\tilde{C}_p + 1.25R)\mu/M$. The calculated results follow, with μ , \tilde{C}_p and k in the units of the problem statement:

T, K	$T/357$	Ω_μ	$10^4 \mu$	$(\tilde{C}_p + 1.25R)$	$10^5 k_{\text{pred}}$	$k_{\text{obs}}/k_{\text{pred}}$
198	0.5546	2.0915	0.8931	10.54	1.33	0.985
275	0.7703	1.8217	1.2044	10.59	1.80	1.056
276	0.7731	1.8239	1.2091	10.59	1.81	1.066
276	0.7731	1.8239	1.2091	10.59	1.81	1.061
363	1.017	1.5799	1.6008	10.81	2.44	1.074
363	1.017	1.5799	1.6008	10.81	2.44	1.070
395	1.106	1.5138	1.7427	10.91	2.68	1.13
453	1.269	1.4172	1.9935	11.02	3.10	1.14
453	1.269	1.4172	1.9935	11.02	3.10	1.10
495	1.3866	1.3609	2.1701	11.095	3.40	1.09
553	1.549	1.2974	2.406	11.17	3.79	1.09
583	1.633	1.2694	2.5249	11.2	3.99	1.11
583	1.633	1.2694	2.5249	11.2	3.99	1.12
676	1.884	1.1994	2.8775	11.32	4.59	1.10
676	1.884	1.1994	2.8775	11.32	4.59	1.07

Ave. = 1.084

The predicted k values exceed the observed values by an average of 8.4% in this temperature range.

9A.10 Thermal conductivity of chlorine-air mixtures.

Numbering chlorine and air as components 1 and 2, respectively, and inserting their given properties into Eq. 1.4-16, we obtain the following coefficients for Eq. 9.3-17:

$$\Phi_{11} = 1; \quad \Phi_{22} = 1;$$

$$\Phi_{12} = \frac{1}{\sqrt{8}} \left(1 + \frac{70.91}{28.97} \right)^{-1/2} \left[1 + \sqrt{\frac{1.351}{1.854}} \left(\frac{28.97}{70.91} \right)^{1/4} \right]^2 = 0.53899;$$

$$\Phi_{21} = \frac{1}{\sqrt{8}} \left(1 + \frac{28.97}{70.91} \right)^{-1/2} \left[1 + \sqrt{\frac{1.854}{1.351}} \left(\frac{70.91}{28.97} \right)^{1/4} \right]^2 = 1.8105$$

Equation 9.3-17 gives, for binary mixtures:

$$k_{\text{mix}} = \frac{x_1 k_1}{x_1 \Phi_{11} + x_2 \Phi_{12}} + \frac{x_2 k_2}{x_2 \Phi_{21} + x_2 \Phi_{22}}$$

Insertion of the coefficients and compositions for this problem gives

At $x_1 = 0.25$,

$$k_{\text{mix}} = \frac{0.25 \times 0.0896}{0.25 + 0.75 \times 0.53899} + \frac{0.75 \times 0.02614}{0.25 \times 1.8105 + 0.75} = 0.0506 \text{ cal/cm}\cdot\text{s}\cdot\text{K}$$

At $x_1 = 0.5$,

$$k_{\text{mix}} = \frac{0.5 \times 0.0896}{0.5 + 0.5 \times 0.53899} + \frac{0.5 \times 0.02614}{0.5 \times 1.8105 + 0.5} = 0.0675 \text{ cal/cm}\cdot\text{s}\cdot\text{K}$$

At $x_1 = 0.75$,

$$k_{\text{mix}} = \frac{0.75 \times 0.0896}{0.75 + 0.25 \times 0.53899} + \frac{0.25 \times 0.02614}{0.75 \times 1.8105 + 0.25} = 0.0800 \text{ cal/cm}\cdot\text{s}\cdot\text{K}$$

9A.11 Thermal conductivity of quartz sand.

a. For spheres ($g_1 = g_2 = g_3 = 1/3$), Eq. 9A.11-2 reduces to

$$\alpha_i = 3/(2 + k_j/k_0)$$

The resulting α_i values from Eq. 9A.11-2 for the water-saturated sand are

$$\alpha_0 = \frac{3}{2 + (k_0/k_0)} = 1; \quad \alpha_1 = \frac{3}{2 + (k_1/k_0)} = 0.183; \quad \alpha_2 = \frac{3}{2 + (k_2/k_0)} = 0.433$$

and Eq. 9A.11-1 then gives

$$k_{\text{eff}} = \frac{(1)(0.427)(0.00142) + (0.183)(0.510)(0.0204) + (0.433)(0.063)(0.0070)}{(1)(0.427) + (0.183)(0.510) + (0.433)(0.063)}$$

which predicts $k_{\text{eff}} = 6.3 \times 10^{-3}$ cal/cm·s·K, vs. 6.2×10^{-3} observed.

For the same sand when completely dry ($k_1/k_0 = 332$, $k_2/k_0 = 114$), Eq. 9A.11-2 for spheres gives

$$\alpha_0 = \frac{3}{2 + 1} = 1; \quad \alpha_1 = \frac{3}{2 + 332} = 0.00898; \quad \alpha_2 = \frac{3}{2 + 114} = 0.0259$$

and Eq. 9A.11-1 with de Vries' correction factor of 1.25 for dry sand gives

$$\frac{k_{\text{eff}}}{1.25} = \frac{(1)(0.427)(0.0000615) + 0.00898(0.510)(0.0204) + (0.0259)(0.063)(0.0070)}{(1)(0.427) + (0.00898)(0.510) + (0.0259)(0.063)}$$

predicting $k_{\text{eff}} = 0.38 \times 10^{-3}$ cal/cm·s·K, vs. 0.58×10^{-3} observed.

For the same sand when water-saturated at 20°C, de Vries' recommended g_j values give

$$\alpha_0 = \frac{1}{3} \left[\frac{2}{1 + 0.125[(1.42/1.42) - 1]} + \frac{1}{1 + 0.750[(1.42/1.42) - 1]} \right] = 1.000$$

$$\alpha_1 = \frac{1}{3} \left[\frac{2}{1 + 0.125[(20.4/1.42) - 1]} + \frac{1}{1 + 0.750[(20.4/1.42) - 1]} \right] = 0.280$$

$$\alpha_2 = \frac{1}{3} \left[\frac{2}{1 + 0.125[(7.0/1.42) - 1]} + \frac{1}{1 + 0.750[(7.0/1.42) - 1]} \right] = 0.532$$

and Eq. 9A.11-1 gives

$$k_{\text{eff}} = \frac{(1)(0.427)(0.00142) + (0.280)(0.510)(0.0204) + (0.532)(0.063)(0.0070)}{(1)(0.427) + (0.280)(0.510) + (0.532)(0.063)}$$

predicting $k_{\text{eff}} = 6.2 \times 10^{-3}$ cal/cm·s·K, in still better agreement with the observed value 6.2×10^{-3} .

For the same sand when completely dry ($k_1/k_0 = 332$, $k_2/k_0 = 114$), Eq. 9A-11-2 with de Vries' recommended g_j values gives

$$\alpha_0 = \frac{1}{3} \left[\frac{2}{1 + 0.125[1 - 1]} + \frac{1}{1 + 0.750[1 - 1]} \right] = 1.000$$

$$\alpha_1 = \frac{1}{3} \left[\frac{2}{1 + 0.125[332 - 1]} + \frac{1}{1 + 0.750[332 - 1]} \right] = 0.0171$$

$$\alpha_2 = \frac{1}{3} \left[\frac{2}{1 + 0.125[114 - 1]} + \frac{1}{1 + 0.750[114 - 1]} \right] = 0.0048$$

and Eq. 9A.11-1 with de Vries' correction factor for dry sand gives

$$\frac{k_{\text{eff}}}{1.25} = \frac{(1)(0.477)(0.00006) + (0.0171)(0.510)(0.0204) + (0.0048)(0.063)(0.0070)}{(1)(0.477) + (0.0171)(0.510) + (0.0048)(0.063)}$$

predicting $k_{\text{eff}} = 0.54 \times 10^{-3}$ cal/cm·s·K, vs. 0.58×10^{-3} observed.

(b) Equation 9.6-1 gives

$$\frac{k_{\text{eff}}}{k_0} = 1 + \frac{3\phi}{\left(\frac{k_1 + 2k_0}{k_1 - k_0} \right) - \phi}$$

Insertion of $\phi = 0.573$ and $k_1 = 0.0189$ cal/cm·s·K for the solids gives, for the water-saturated sand,

$$\frac{k_{\text{eff}}}{k_0} = 1 + \frac{3(0.573)}{\left(\frac{0.0189 + (2)(0.00142)}{0.0189 - 0.00142} \right) - 0.573} = 3.563$$

predicting $k_{\text{eff}} = 5.1 \times 10^{-3}$ cal/cm·s·K, vs. 6.2×10^{-3} observed. This is not as good as the prediction in (a) from Eq. 9A.6-11 with de Vries' g_j values.

For the completely dry sand, insertion of the k value for air as k_0 into Eq. 9.6-1 gives

$$\frac{k_{\text{eff}}}{k_0} = 1 + \frac{3(0.573)}{\left(\frac{0.0189 + 2(0.0000615)}{0.0189 - 0.0000615} \right) - 0.573} = 4.935$$

predicting $k_{\text{eff}} = 0.30 \times 10^{-3}$ cal/cm·s·K, vs. 0.58×10^{-3} observed. The result in (a), from Eq. 9A.6-11 with de Vries' g_j , is better.

Predictions of k_{eff} are more difficult for dry sand than for water-saturated sand. An oblate-spheroidal model gives little advantage according to the present data.

9A.12 Calculation of molecular diameters from transport properties.

a. Equation 1.4-9 and the viscosity value from Problem 9A.2 yield the following molecular diameter calculation for Argon in cgs units:

$$\begin{aligned} d &= \sqrt{2/3\mu} \left(\frac{MKT}{\tilde{N}\pi^3} \right)^{1/4} \\ &= \sqrt{2/3/(2.278 \times 10^{-4})} \left(\frac{39.948 \times 1.38066 \times 10^{-16} \times 300}{6.02214 \times 10^{23} \times \pi^3} \right)^{1/4} \\ &= 2.95 \times 10^{-8} \text{ cm} \end{aligned}$$

b. Equation 9.3-12 and the k value from Problem 9A.2 give the following molecular diameter calculation for Argon in cgs units:

$$\begin{aligned} d &= \sqrt{\frac{1}{k}} \left(\frac{K^3 T \tilde{N}}{\pi^3 M} \right)^{1/4} \\ &= \sqrt{\frac{1}{0.01784 \times 10^5}} \left(\frac{(1.38066 \times 10^{-16})^3 \times 300 \times 6.02214 \times 10^{23}}{\pi^3 \times 39.948} \right)^{1/4} \\ &= 1.86 \times 10^{-8} \text{ cm} \end{aligned}$$

c. Equation 1.4-14, Tables E.1 and E.2 and the viscosity from Problem 9A.2 give for Argon in cgs units,

$$\begin{aligned} \sigma &= \sqrt{\frac{5}{16\mu\Omega_\mu}} \left(\frac{MKT}{\tilde{N}\pi} \right)^{1/4} \\ &= \sqrt{\frac{5}{16 \times 2.278 \times 10^{-4} \times 1.1000}} \left(\frac{39.948 \times 1.38066 \times 10^{-16} \times 300}{6.02214 \times 10^{23} \times \pi} \right)^{1/4} \\ &= 3.415 \times 10^{-8} \text{ cm} \end{aligned}$$

Equation 9.3-15, Tables E.1 and E.2 and the k value from Problem 9A.2 give for Argon in cgs units,

$$\begin{aligned} \sigma &= \sqrt{\frac{75}{64k\Omega_k}} \left(\frac{K^3 T \tilde{N}}{\pi M} \right)^{1/4} \\ &= \sqrt{\frac{75}{64 \times 1784 \times 1.100}} \left(\frac{(1.38066 \times 10^{-16})^3 \times 300 \times 6.02214 \times 10^{23}}{\pi \times 39.948} \right)^{1/4} \\ &= 3.409 \times 10^{-8} \text{ cm} \end{aligned}$$

d. The excellent agreement between the results for σ , and the poor agreement for d , show that the data are represented much better by the Chapman-Enskog theory than by the simple hard-sphere kinetic theory.

9C.1 Enskog theory for dense gases

a. Equation 9C.1-4 can be written with pressure and temperature as the independent variables, thus:

$$y = \frac{\tilde{V}}{R} \left(- \frac{(\partial \tilde{V} / \partial T)_p}{(\partial \tilde{V} / \partial p)_T} \right) - 1$$

We may now use $\tilde{V} = ZRT/p$ and rewrite the derivatives appearing above as

$$\left(\frac{\partial \tilde{V}}{\partial T} \right)_p = \frac{R}{p} \left(\frac{\partial}{\partial T} ZT \right)_p = \frac{R}{p} \left(Z + T \left(\frac{\partial Z}{\partial T} \right)_p \right) = \frac{RZ}{p} \left(1 + \left(\frac{\partial \ln Z}{\partial \ln T} \right)_p \right)$$

$$\left(\frac{\partial \tilde{V}}{\partial p} \right)_T = RT \left(\frac{\partial}{\partial p} \frac{Z}{p} \right)_T = - \frac{RT}{p^2} \left(Z - p \left(\frac{\partial Z}{\partial p} \right)_T \right) = - \frac{RTZ}{p^2} \left(1 - \left(\frac{\partial \ln Z}{\partial \ln p} \right)_T \right)$$

When these expressions are substituted into the expression for y , we get

$$y = Z \left[\frac{1 + (\partial \ln Z / \partial \ln T)_p}{1 - (\partial \ln Z / \partial \ln p)_T} \right] - 1 \quad \text{or} \quad y = Z \left[\frac{1 + (\partial \ln Z / \partial \ln T_r)_{p_r}}{1 - (\partial \ln Z / \partial \ln p_r)_{T_r}} \right] - 1$$

b. First one would differentiate the Hougen-Watson Z chart to get the derivatives appearing in the result in (a) and hence $y(p_r, T_r)$. Then for a given reduced temperature and pressure, one would calculate the right sides of Eqs. 9C.1-1 and 2; call these quantities $f_\mu(p_r, T_r)$ and $f_k(p_r, T_r)$. Then

$$\frac{k}{k^\circ} = \frac{\mu}{\mu^\circ} \cdot \frac{f_k(p_r, T_r)}{f_\mu(p_r, T_r)}$$

One can then read off values of μ from the Uyehara-Watson chart and construct a chart for the thermal conductivity. This procedure is not recommended for polyatomic gases.

10A.1 Heat loss from an insulated pipe.

We use the notation of Fig. 10.6-2. When the temperatures at the inner and outer surfaces are known, Eq. 10.6-29 can be reduced to

$$\frac{Q_0}{L} = \frac{2\pi L(T_0 - T_3)}{\left[\frac{\ln(r_1/r_0)}{k_{01}} + \frac{\ln(r_2/r_1)}{k_{12}} + \frac{\ln(r_3/r_2)}{k_{23}} \right]}$$

The r_i for this problem are:

$$r_0 = 2.067/2 = 1.0335 \text{ in}$$

$$r_1 = 1.0335 + 0.154 = 1.19 \text{ in}$$

$$r_2 = 1.19 + 2 = 3.19$$

$$r_3 = 3.19 + 2 = 5.19$$

Insertion of numerical values into the above formula gives:

$$\begin{aligned} \frac{Q_0}{L} &= \frac{2\pi(250 - 90 \text{ F})}{\left[\frac{\ln(1.19/1.0335)}{26.1} + \frac{\ln(3.19/1.19)}{0.04} + \frac{\ln(5.19/3.19)}{0.03} \right] \text{ F}\cdot\text{hr}\cdot\text{ft}/\text{Btu}} \\ &= \frac{320\pi}{0.0054 + 24.7 + 16.2} = 24 \text{ Btu/hr per foot of pipe} \end{aligned}$$

10A.2 Heat loss from a rectangular fin.

From Eq. 10.7-14 we obtain the heat loss expression

$$Q = 2WLh(T_w - T_a) \cdot \eta$$

in which η is given by Eq. 10.7-16:

$$\eta = \frac{\tanh N}{N} \quad \text{with } N = \sqrt{\frac{hL^2}{kB}}$$

For the conditions of this problem,

$$N = \sqrt{\frac{hL^2}{kB}} = \sqrt{\frac{(120 \text{ Btu/hr}\cdot\text{ft}^2\cdot\text{F})(0.2 \text{ ft})^2}{(60 \text{ Btu/hr}\cdot\text{ft}\cdot\text{F})(0.08/12 \text{ ft})}} = \sqrt{12} = 3.4641$$

and

$$\eta = \tanh(3.4641)/3.4641 = 0.2881$$

The foregoing heat loss expression then gives:

$$\begin{aligned} Q &= 2WLh(T_0 - T_a) \cdot \eta \\ &= 2(1.0 \text{ ft})(0.2 \text{ ft})(120 \text{ Btu/hr}\cdot\text{ft}\cdot\text{F})(500 - 350 \text{ F})(0.2881) \\ &= 2074 \text{ Btu/hr} \end{aligned}$$

10A.3 Maximum temperature in a lubricant.

The parallel-plate approximation in §10.4 is used here to estimate the temperature rise; more accurate results will be presented in Chapter 11. Multiplication of Eq. 10.4-9 by $(T_b - T_0)$, and setting $T_b = T_0$, gives the temperature profile

$$T - T_0 = \frac{1}{2} \frac{\mu v_b^2}{k} (x/b)[1 - (x/b)]$$

The maximum temperature occurs at $x = b/2$; hence, with $v_b = \Omega R$,

$$\begin{aligned} T_{\max} - T_0 &= \frac{1}{8} \frac{\mu v_b^2}{k} = \frac{1}{8} \frac{\mu \Omega^2 R^2}{k} \\ &= \frac{1}{8} \frac{(0.923 \text{ g/cm}\cdot\text{s})(7908 \times 2\pi/60 \text{ radians/s})^2 (5.06 \text{ cm})^2}{(0.0055 \text{ cal/s}\cdot\text{cm}\cdot\text{C})(4.1840 \times 10^7 \text{ g}\cdot\text{cm}^2/\text{s}^2/\text{cal})} \\ &= 8.8 \text{ C} = 16 \text{ F} \end{aligned}$$

Thus, the maximum temperature in the oil is $T_{\max} = 158 + 16 = 174^\circ\text{F}$.

10A.4 Current-carrying capacity of a wire.

Consider a straight cylindrical wire of radius $r_2 = 0.040$ in, tightly covered with plastic insulation of outer radius $r_3 = 0.12/2 = 0.06$ in. Since the heat generated by electrical dissipation in the wire must flow radially outward, the maximum temperature in the plastic will occur at the wire-plastic interface $r = r_2$. The rate of heat loss from the wire reaches its largest permitted value when $T = 200^\circ\text{F}$ at $r = r_2$ while the ambient temperature T_4 is 100°F . From Eq. 10.6-29 we calculate this rate as

$$\begin{aligned}\frac{Q_{\max}}{L} &= 2\pi(T_{2,\max} - T_4) / \left[\frac{\ln(r_3/r_2)}{k_{23}} + \frac{1}{r_3 h_3} \right] \\ &= 2\pi(200 - 100 \text{ F}) / \left[\frac{\ln(0.060/0.020)}{0.20} + \frac{1}{(0.060/12)(1.5)} \text{ hr}\cdot\text{ft}\cdot\text{F}/\text{Btu} \right] \\ &= 200\pi / [5.5 + 133] = 4.54 \text{ Btu}/\text{hr}\cdot\text{ft} \\ &= 0.0436 \text{ watt}/\text{cm}\end{aligned}$$

Next, we equate this rate of heat loss to the electrical energy dissipation:

$$Q_{\max}/L = i_{\max}^2 R_e/L$$

in which

$$\begin{aligned}R_e/L &= (\pi r_2^2 k_e)^{-1} = \text{wire resistance } R_e \text{ per unit length} \\ &= (\pi(0.02 \times 2.54 \text{ cm})^2 (5.1 \times 10^5 \text{ ohm}^{-1}\text{cm}^{-1}))^{-1} \\ &= 2.42 \times 10^{-4} \text{ ohm}/\text{cm} \\ i_{\max} &= \text{maximum permitted current}\end{aligned}$$

The current-carrying capacity of the wire is then

$$\begin{aligned}i_{\max} &= \sqrt{(Q_{\max}/L)/(R_e/L)} \\ &= \sqrt{(0.0435 \text{ watts}/\text{cm})/(2.42 \times 10^{-4} \text{ ohm}/\text{cm})} \\ &= 13.4 \text{ amperes}\end{aligned}$$

10A.5 Free convection velocity.

a. The solution for v_z in Eq. 10.9-15 has the form

$$v_z = A [u^3 - u] \quad \text{in which } A = \frac{(\bar{\rho}g\bar{\beta}\Delta T)B^2}{12\mu} \quad \text{and} \quad u = (y/B),$$

and its average over the upward-moving stream is

$$\begin{aligned} \langle v_z^{(\text{up})} \rangle &= A \int_{u=-1}^{u=0} [u^3 - u] du = \frac{A}{4} \\ &= \frac{(\bar{\rho}g\bar{\beta}\Delta T)B^2}{48\mu} \end{aligned}$$

in agreement with Eq. 10.9-16.

b. For the conditions of this problem,

$$\begin{aligned} \bar{T} &= \frac{1}{2}(T_1 + T_2) = 60 \text{ }^\circ\text{C} = 333.2 \text{ K;} \\ \bar{\beta} &= 1/\bar{T} = 3 \times 10^{-3} \text{ K}^{-1}. \end{aligned}$$

c. Insertion of known values into Eq. 10.9-16 (with $\mu/\bar{\rho}$ averaged as $\mu(\bar{T})/\rho(\bar{T}) = \nu(\bar{T}) = 0.1886 \text{ cm}^2/\text{s}$) then gives:

$$\begin{aligned} \langle v_z^{(\text{up})} \rangle &= \frac{(980.7 \text{ cm/s}^2)((3 \times 10^{-3} \text{ K}^{-1})(80 \text{ C})(0.3 \text{ cm})^2}{(48)(0.1886 \text{ cm}^2/\text{s})} \\ &= 2.3 \text{ cm/s} \end{aligned}$$

10A.6 Insulating power of a wall.

a. Application of Eq. 10.6-9 to the data for the plastic panel gives

$$\begin{aligned} q_0 &= \frac{k_{12}(T_1 - T_2)}{(x_2 - x_1)} = \frac{(0.075 \text{ Btu/hr}\cdot\text{ft}\cdot\text{F})(69 - 61 \text{ F})}{(0.502/12 \text{ ft})} \\ &= 14.3 \text{ Btu/hr}\cdot\text{ft}^2 \end{aligned}$$

b. The thermal resistance of the wall is therefore

$$\begin{aligned} R_{23} &= \frac{(T_2 - T_3)}{q_0} = \frac{(61 - 0 \text{ F})}{14.3 \text{ Btu/hr}\cdot\text{ft}^2} \\ &= 4.2 \frac{\text{F}}{\text{Btu/hr}\cdot\text{ft}^2} \end{aligned}$$

10A.7 Viscous heating in a ball-point pen.

The parallel-plate approximation in §10.4 is used here to estimate the viscous heating of the fluid; more accurate results will be presented in Chapter 11. Multiplication of Eq. 10.4-9 by $(T_b - T_0)$, and setting $T_b = T_0$, gives the temperature profile in the ink,

$$T - T_0 = \frac{1}{2} \frac{\mu v_b^2}{k} (x/b)[1 - (x/b)]$$

valid when both adjoining surfaces are at temperature T_0 .

At $x = b/2$ the temperature rise attains its maximum value

$$(T - T_0)_{\max} = \frac{1}{8} \frac{\mu v_b^2}{k}$$

Insertion of the data for this problem gives

$$\begin{aligned} (T - T_0)_{\max} &= \frac{1}{8} \frac{(10^4 \times 0.01 \text{ g/cm}\cdot\text{s})(100 \times 2.54/60 \text{ cm/s})^2}{(5 \times 10^{-4} \times 4.1840 \times 10^7 \text{ g}\cdot\text{cm/s}^3\cdot\text{K})} \\ &= 0.011 \text{ K} \end{aligned}$$

as the maximum dissipative temperature rise in the ink. Thus, the warming of the ink by viscous dissipation will be negligible compared with the warming of the pen by contact with the hand of the user.

10A.8 Temperature rise in an electrical wire.

The maximum temperature in the wire occurs at the centerline. Equation 10.2-23 gives this value as

$$T_{\max} = T_{\text{air}} + \frac{S_e R^2}{4k} + \frac{S_e R}{2h}$$

in which

$$S_e = \frac{I^2}{k_e} = \left(\frac{k_e E}{L} \right)^2 \frac{1}{k_e} = \left(\frac{E^2 k_e}{L^2} \right)$$

The Wiedemann-Franz-Lorenz relation of Eq. 9.5-1, with the Lorenz constant for copper, gives

$$\frac{k}{k_e T} = 223 \times 10^{-8} \text{ volt}^2/\text{K}^2$$

and k for copper ranges from 384.1 W/m·K at 25°C to 379.9 W/m·K at 100°C. Assuming the temperature rise to be small, we evaluate the conductivity values at 25°C=298.15K:

$$\begin{aligned} k &= 384.1 \text{ W/m}\cdot\text{K} \\ k_e &= \frac{k}{(\text{Lorenz constant})(T)} \\ &= \frac{(384.1 \text{ W/m}\cdot\text{K})}{(223 \times 10^{-8} \text{ volt}^2/\text{K})(298.15 \text{ K})} \\ &= 5.78 \times 10^5 \text{ ohm}^{-1}\text{m}^{-1} \end{aligned}$$

We can then calculate S_e as

$$\begin{aligned} S_e &= \left(\frac{E^2 k_e}{L^2} \right) \\ &= \frac{(0.6 \text{ volt})^2 (5.78 \times 10^5 \text{ ohm}^{-1}\text{m}^{-1})}{(15 \text{ ft} \times 0.3048 \text{ m/ft})^2} \\ &= 9.95 \times 10^3 \text{ W/m}^3 \end{aligned}$$

and the maximum temperature elevation in the wire as

$$\begin{aligned} T_{\max} - T_{\text{air}} &= \frac{(9.95 \times 10^3 \text{ W/m}^3)(0.005 \text{ m})^2}{4(384.1 \text{ W/m}\cdot\text{K})} + \frac{(9.95 \times 10^3 \text{ W/m}^3)(0.005 \text{ m})}{2([5.7 \times 5.6782] \text{ W/m}^2\cdot\text{K})} \\ &= 1.6 \times 10^{-4} + 0.77 \approx 0.77 \text{ K} \end{aligned}$$

b. The temperature difference across the wire is 4 orders of magnitude smaller than that in the surrounding air, as shown by the terms in the last line of the calculation.

10B.1 Heat conduction from a sphere to a stagnant fluid

a. The heat is being conducted in the r direction only. Therefore, we select a shell of thickness Δr over which we make the energy balance:

$$\Delta r q_r 4\pi r^2 q_r \Big|_r - 4\pi (r + \Delta r)^2 q_r \Big|_{r+\Delta r} = 0 \quad \text{or} \quad 4\pi (r^2 q_r) \Big|_r - 4\pi (r^2 q_r) \Big|_{r+\Delta r} = 0$$

We now divide by Δr and then take the limit as Δr goes to zero

$$\lim_{\Delta r \rightarrow 0} \frac{(r^2 q_r) \Big|_{r+\Delta r} - (r^2 q_r) \Big|_r}{\Delta r} = 0$$

We then use the definition of the first derivative to get

$$\frac{d}{dr}(r^2 q_r) = 0 \quad \text{and} \quad \frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0$$

In the second equation we have inserted Fourier's law of heat conduction with constant thermal conductivity.

b. Integration of this equation twice with respect to r gives

$$r^2 \frac{dT}{dr} = C_1 \quad \text{and} \quad T = -\frac{C_1}{r} + C_2$$

The boundary conditions then gives $C_1 = -R(T_R - T_\infty)$, $C_2 = T_\infty$, and

$$\frac{T - T_\infty}{T_R - T_\infty} = \frac{R}{r}$$

c. The heat flux at the surface is

$$q_r \Big|_{r=R} = -k \frac{dT}{dr} \Big|_{r=R} = +kR(T_R - T_\infty) \frac{1}{r^2} \Big|_{r=R} = \frac{k(T_R - T_\infty)}{R} \equiv h(T_R - T_\infty)$$

so that $h = k/R = 2k/D$ and $Nu = 2$.

d. Bi contains k of the solid; Nu contains k of the fluid.

10B.2 Viscous heating in slit flow

Equations 10.4-5 and 10.4-6 are still valid for this problem. Since at $x = b$, $q_x = -k(dT/dx) = 0$, we get from Eq. 10.4-5:

$$-\mu b \left(\frac{v_b}{b} \right)^2 = C_1 \quad \text{or} \quad C_1 = -\frac{\mu v_b^2}{b}$$

And from Eq. 10.4-6, we get $C_2 = T_0$.

Then substitution of these expressions for the integration constants into Eq. 10.4-6 yields

$$T = -\left(\frac{\mu}{k} \right) \left(\frac{v_b}{b} \right)^2 \frac{x^2}{2} + \frac{\mu v_b^2}{bk} x + T_0$$

When this is rearranged in dimensionless form we have

$$\frac{T - T_0}{\mu v_b^2 / k} = \left(\frac{x}{b} \right) - \frac{1}{2} \left(\frac{x}{b} \right)^2$$

10B.3 Heat conduction in a nuclear fuel rod assembly

The differential equation may be set up following the procedure in §10.2, by replacing S_e by S_n in Eq. 10.2-6. Then, when Fourier's law with constant thermal conductivity is substituted into the thus modified Eq. 10.2-6, we get

$$-k_F \frac{d}{dr} \left(r \frac{dT_F}{dr} \right) = S_{n0} \left[1 + b \left(\frac{r}{R_F} \right)^2 \right] r$$

for the heat conduction equation in the fuel rod. In the cladding a similar equation, without the source term, is appropriate:

$$-k_C \frac{d}{dr} \left(r \frac{dT_C}{dr} \right) = 0$$

The boundary conditions in this problem are

- B. C. 1: At $r = 0$, T_F is finite
- B. C. 2: At $r = R_F$, $T_F = T_C$
- B. C. 3: At $r = R_F$, $-k_F(dT_F/dr) = -k_C(dT_C/dr)$
- B. C. 4: At $r = R_C$, $-k_C(dT_C/dr) = h_L(T_C - T_L)$

Integrating the above differential equations twice gives

$$\frac{dT_F}{dr} = -\frac{S_{n0}r}{2k_F} \left(1 + \frac{b}{R_F^2} \frac{r^2}{2} \right) + \frac{C_1}{r} \qquad \frac{dT_C}{dr} = \frac{C_3}{r}$$

$$T_F = -\frac{S_{n0}r^2}{4k_F} \left(1 + \frac{b}{R_F^2} \frac{r^2}{4} \right) + C_1 \ln r + C_2; \quad T_C = C_3 \ln r + C_4$$

The constant is zero by B. C. 1, since the temperature is not infinite at the axis of the fuel rod. From B. C. 3, we can find C_3 :

$$C_3 = -\frac{S_{n0}R_F^2}{2k_C} \left(1 + \frac{b}{2} \right)$$

From B. C. 4, we get C_4 :

$$C_4 = T_L - \left(\frac{k_C}{R_C h_L} + \ln R_C \right) C_3 = T_L + \left(\frac{k_C}{R_C h_L} + \ln R_C \right) \frac{S_{n0} R_F^2}{2k_C} \left(1 + \frac{b}{2} \right)$$

And finally C_2 can be obtained from B. C. 2:

$$C_2 = T_L + \frac{S_{n0} R_F^2}{4k_F} \left(1 + \frac{b}{4} \right) + \frac{S_{n0} R_F^2}{2k_C} \left(1 + \frac{b}{2} \right) \left(\ln \frac{R_C}{R_F} + \frac{k_C}{R_C h_L} \right)$$

Then we can get the maximum temperature at the axis of the fuel rod:

$$T_{F,\max} = T_L + \frac{S_{n0} R_F^2}{4k_F} \left(1 + \frac{b}{4} \right) + \frac{S_{n0} R_F^2}{2k_C} \left(1 + \frac{b}{2} \right) \left(\ln \frac{R_C}{R_F} + \frac{k_C}{R_C h_L} \right)$$

10B.4 Heat conduction in an annulus

a. The energy balance on a cylindrical shell of thickness Δr and length L is

$$2\pi r L q_r \Big|_r - 2\pi(r + \Delta r)L q_r \Big|_{r+\Delta r} = 0 \quad \text{or} \quad 2\pi L (r q_r) \Big|_r - 2\pi L (r q_r) \Big|_{r+\Delta r} = 0$$

When this equation is divided by $2\pi L$ and the limit is taken as Δr goes to zero, we get

$$\frac{d}{dr}(r q_r) = 0$$

which may be integrated to give

$$r q_r = C_1 \quad \text{or} \quad -k \frac{dT}{dr} = \frac{C_1}{r}$$

The thermal conductivity varies linearly with temperature, so that

$$k = k_0 + (k_1 - k_0) \left(\frac{T - T_0}{T_1 - T_0} \right) \equiv k_0 + (k_1 - k_0) \Theta$$

Then

$$-[k_0 + (k_1 - k_0) \Theta] \frac{dT}{dr} = \frac{C_1}{r} \quad \text{or} \quad -(T_1 - T_0) [k_0 + (k_1 - k_0) \Theta] \frac{d\Theta}{dr} = \frac{C_1}{r}$$

This first-order, separable differential equation may be integrated:

$$-(T_1 - T_0) \left[k_0 + \frac{1}{2}(k_1 - k_0) \Theta \right] \Theta = C_1 \ln r + C_2$$

The constants of integration may be found from the boundary conditions: $\Theta(r_0) = 0$ and $\Theta(r_1) = 1$.

$$0 = C_1 \ln r_0 + C_2 \quad \text{and} \quad -(T_1 - T_0) \left[k_0 + \frac{1}{2}(k_1 - k_0) \right] = C_1 \ln r_1 + C_2$$

When these relations are subtracted, and equation for C_1 is obtained:

$$C_1 = -\frac{(T_1 - T_0)}{\ln(r_1/r_0)} \left[\frac{1}{2}(k_0 + k_1) \right]$$

and C_2 may also be obtained if desired.

The heat flow through the wall may then be obtained:

$$Q = 2\pi r_0 L q_r|_{r=r_0} = 2\pi L r_0 \left(\frac{C_1}{r_0} \right) = 2\pi L \frac{(T_0 - T_1)}{\ln(r_1/r_0)} \left[\frac{1}{2}(k_0 + k_1) \right]$$

b. Let the ratio of the outer to the inner radius be written as $r_1/r_0 = 1 + \varepsilon$, where ε is very small. Then use the Taylor series for the logarithm as given in Eq. C.2-3: $\ln(1 + \varepsilon) = \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 - \dots$. If we keep just one term of the series, then this corresponds to

$$\varepsilon = (r_1/r_0) - 1 = (r_1 - r_0)/r_0$$

When this is substituted into the expression for Q we get

$$Q = 2\pi L r_0 \left[\frac{1}{2}(k_0 + k_1) \right] \frac{T_0 - T_1}{r_1 - r_0}$$

This is just: area times average thermal conductivity times a temperature gradient.

10B.5 Viscous heat generation in a polymer melt

We can start with a modification of Eq. 10.4-4:

$$-k \frac{dT}{dx} - v_z \left(m \left| \frac{dv_z}{dx} \right|^{n-1} \frac{dv_z}{dx} \right) = C_1$$

Since the velocity gradient is positive in this problem, the absolute value operation is not needed:

$$-k \frac{dT}{dx} - m v_z \left(\frac{dv_z}{dx} \right)^n = C_1$$

When the linear velocity profile is inserted, this becomes

$$-k \frac{dT}{dx} - m x \left(\frac{v_b}{b} \right)^{n+1} = C_1$$

Integration with respect to x gives

$$-kT = C_1 x + \frac{1}{2} m x^2 \left(\frac{v_b}{b} \right)^{n+1} + C_2$$

The constants are then determined by the boundary conditions given in Eqs. 10.4-7 and 8, and final result is

$$\frac{T - T_0}{T_b - T_0} = \frac{x}{b} + \frac{1}{2} \frac{m b^2}{k(T_b - T_0)} \left(\frac{v_b}{b} \right)^{n+1} \left[\left(\frac{x}{b} \right) - \left(\frac{x}{b} \right)^2 \right]$$

or

$$\frac{T - T_0}{T_b - T_0} = \frac{x}{b} + \frac{1}{2} \text{Br}_n \frac{x}{b} \left(1 - \frac{x}{b} \right)$$

which should be compared to the Newtonian result in Eq. 10.4-9.

10B.6 Insulation thickness for a furnace wall

Let the regions be labeled as follows:

Refractory brick	"01"
Insulating brick	"12"
Steel	"23"

and we may use the formulas given in Eqs. 10.6-8, 9, and 10.

The minimum wall thickness will occur when $T_1 = 2000^\circ\text{F}$. If for the sake of being on the safe side, let $T_0 = 2500^\circ\text{F}$. Then for the region "01" the thickness must be

$$x_1 - x_0 = \frac{k_{01}(T_0 - T_1)}{q_0} = \frac{\frac{1}{2}(4.1 + 3.6)(2500 - 2000)}{5000} = 0.39\text{ft}$$

Here we have taken the thermal conductivity of the refractory brick to be the arithmetic average of the values the thermal conductivity at 2000°F and 2500°F (the latter estimated by linear extrapolation from the given data).

For the remaining two regions, we may add Eqs. 10.6-9 and 10 to get

$$T_1 - T_3 = q_0 \left(\frac{x_2 - x_1}{k_{12}} + \frac{x_3 - x_2}{k_{23}} \right)$$

or, taking the steel temperature to be 100,

$$2000 - 100 = 5000 \left(\frac{x_2 - x_1}{\frac{1}{2}(0.9 + 1.8)} + \frac{(0.25)\frac{1}{12}}{26.1} \right)$$

This gives $x_2 - x_1 = 0.51\text{ft}$.

10B.7 Forced-convection heat transfer in flow between parallel plates

a. Since the temperature depends on both x and z , we make an energy balance over a region of volume $W\Delta x\Delta z$, in which W is the dimension of the slit in the y direction. The various contributions to the energy balance are:

$$\text{Total energy in at } x: \quad e_x|_x W\Delta z$$

$$\text{Total energy out at } x + \Delta x: \quad e_x|_{x+\Delta x} W\Delta z$$

$$\text{Total energy in at } z: \quad e_z|_z W\Delta x$$

$$\text{Total energy out at } z + \Delta z: \quad e_z|_{z+\Delta z} W\Delta x$$

$$\text{Work done on fluid by gravity:} \quad \rho v_z g_z W\Delta x\Delta z$$

When these terms are added together and divided by $W\Delta x\Delta z$, we get

$$-\frac{\partial e_x}{\partial x} - \frac{\partial e_z}{\partial z} - \rho v_z g = 0$$

since gravity is acting in the $-z$ direction.

Now we use Eqs. 9.8-6 and 9.8-8 to write out the x and z components of the combined energy flux:

$$e_x = \tau_{xz} v_z + q_x = -\left(\mu \frac{\partial v_z}{\partial x}\right) v_z - k \frac{\partial T}{\partial x}$$

$$\begin{aligned} e_z &= \left(\frac{1}{2}\rho v_z^2\right) v_z + \rho \hat{H} v_z + \tau_{zz} v_z + q_z \\ &= \left(\frac{1}{2}\rho v_z^2\right) v_z + (p - p^0) v_z + \rho \hat{C}_p (T - T^0) v_z - \left(2\mu \frac{\partial v_z}{\partial z}\right) v_z - k \frac{\partial T}{\partial z} \end{aligned}$$

Substituting these expressions into the energy balance, and making use of the fact that v_z depends only on x gives

$$\rho \hat{C}_p v_z \frac{\partial T}{\partial z} = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) + \mu \left(\frac{\partial v_z}{\partial x} \right)^2 + \left(-\frac{\partial p}{\partial z} + \mu \frac{\partial^2 v_z}{\partial x^2} - \rho g \right)$$

The term in the last parentheses is zero by the equation of motion, the term just before that is the viscous heating (which we neglect), and in the first parentheses we neglect the heat conduction in the z-direction.

b. Hence we get

$$\rho \hat{C}_p v_{z,\max} \left[1 - \left(\frac{x}{B} \right)^2 \right] \frac{\partial T}{\partial z} = k \frac{\partial^2 T}{\partial x^2} \quad \text{or} \quad (1 - \sigma^2) \frac{\partial \Theta}{\partial \zeta} = \frac{\partial^2 \Theta}{\partial \sigma^2}$$

with the boundary and initial conditions: at $\sigma = \pm 1$, $\pm(\partial \Theta / \partial \sigma) = 1$, and at $\zeta = 0$, $\Theta = 0$.

c. For large z we propose the solution $\Theta(\sigma, \zeta) = C_0 \zeta + \Psi(\sigma)$. Then $\Psi(\sigma)$ has to satisfy the ordinary differential equation

$$\frac{\partial^2 \Psi}{\partial \sigma^2} = C_0 (1 - \sigma^2)$$

which is easily integrated. The expression for $\Theta(\sigma, \zeta)$ is then

$$\Theta(\sigma, \zeta) = C_0 \zeta + C_0 \left(\frac{1}{2} \sigma^2 - \frac{1}{12} \sigma^4 \right) + C_1 \sigma + C_2$$

Application of the boundary conditions at $\sigma = \pm 1$ gives $C_1 = 0$ and $C_0 = \frac{3}{2}$. The remaining constant has to be obtained from an integral condition:

$$\zeta = \int_0^1 \Theta(\sigma, \zeta) (1 - \sigma^2) d\sigma$$

This gives $C_2 = -\frac{39}{280}$. Combining these results we get

$$\Theta(\sigma, \zeta) = \frac{3}{2} \zeta + \frac{3}{2} \left(\frac{1}{2} \sigma^2 - \frac{1}{12} \sigma^4 \right) - \frac{39}{280}$$

which is in accordance with Eq. 10B.7-4.

10B.8 Electrical heating of a pipe

For the assumptions made in the problem statement we may assume that temperature is a function only of r and that

$$q_r|_{\kappa R} = -k \frac{\partial T}{\partial r} \Big|_{\kappa R} = 0$$

therefore that all generated heat must leave from the surface at R by Newton's law of

cooling:

$$VS_e = Ah(T_1 - T_s)$$

or

$$T|_R = T_1 = T_s + R(1 - \kappa)S_e / 2h$$

These relations provide a basis for setting up a differential equation as well as giving the boundary conditions for integrating it. The next step is a shell balance over the region between any radial position " r " and a nearby position " $r + \Delta r$ ":

$$2\pi r L q_r|_r + 2\pi r L S_e \Delta r = 2\pi r L q_r|_{r+\Delta r}$$

or

$$\frac{1}{r} \left[\frac{r q_r|_{r+\Delta r} - r q_r|_r}{\Delta r} \right] \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} r q_r = S_e$$

It follows, just as for Ex. 10.2, that

$$q_r = -k \frac{\partial T}{\partial r} = \frac{r S_e}{2} + \frac{C_1}{r}$$

However, the above boundary condition now gives

$$C_1 = -\frac{(\kappa R)^2 S_e}{2}$$

and

$$\frac{\partial T}{\partial r} = -\frac{r S_e}{2k} + \frac{1}{r} \frac{(\kappa R)^2 S_e}{2k}$$

$$T = -\frac{r^2 S_e}{4k} + \ln r \cdot \frac{(\kappa R)^2 S_e}{2k} + C_2$$

Therefore

$$T = T_1 + \frac{R^2 S_e}{4k} \left\{ \left[1 - (r/R)^2 \right] - 2\kappa^2 \ln(R/r) \right\}$$

and T_1 is related to the surrounding temperature T_s by the above equation.

10B.9 Plug flow with forced-convection heat transfer

a. We can adapt the solution in §10.8, by replacing $v_{z,\max}$ by v_0 , and omitting the factor $[1 - (r/R)^2]$ in Eq. 10.8-12. Then we can give the results corresponding to those in §10.8 as follows:

$$(10.8-25) \quad \zeta = \int_0^1 \Theta(\xi, \zeta) \xi d\xi$$

$$(10.8-26) \quad \frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{d\Psi}{d\xi} \right) = C_0$$

$$(10.8-27) \quad \Theta(\xi, \zeta) = C_0 \zeta + \frac{1}{4} C_0 \xi^2 + C_1 \ln \xi + C_2$$

$$(10.8-28, 29, 30) \quad C_1 = 0; C_0 = 2; C_2 = -\frac{1}{4}$$

With these coefficients, Eq. 10B.9-1 follows.

b. The solution of Problem 10B.7 can be adapted by replacing $v_{z,\max}$ by v_0 , and omitting the factor $[1 - (x/B)^2]$ in the energy equation. The results, analogous to those in part (a) are

$$(10.8-25) \quad \zeta = \int_0^1 \Theta(\sigma, \zeta) \sigma d\sigma$$

$$(10.8-26) \quad \frac{d^2\Psi}{d\sigma^2} = C_0$$

$$(10.8-27) \quad \Theta(\sigma, \zeta) = C_0 \zeta + \frac{1}{4} C_0 \sigma^2 + C_1 \sigma + C_2$$

$$(10.8-28, 29, 30) \quad C_1 = 0; C_0 = 1; C_2 = -\frac{1}{6}$$

With these coefficients, one obtains Eq. 10B.9-2.

10B.10 Free convection in an annulus of finite height

a. The appropriate simplification of the energy equation is

$$0 = k \frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right)$$

which has the solution $T = C_1 \ln r + C_2$. When the constants are evaluated using the boundary conditions we get Eq. 10B.10-1.

b. We make use of the linear approximation for the density as a function of temperature:

$$\rho = \rho_1 - \rho_1 \beta_1 (T - T_1)$$

This leads to the equation of motion

$$0 = \mu \frac{1}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) - \left(\frac{dp}{dz} + \rho_1 g \right) + \rho_1 g \beta_1 (T - T_1)$$

Substitution of the expression for the temperature distribution from Eq. 10B.10-1 into the equation of motion, and multiplication by R^2/μ gives

$$\frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{dv_z}{d\xi} \right) = \frac{R^2}{\mu} \left(\frac{dp}{dz} + \rho_1 g \right) + \left[\frac{\rho_1 g \beta_1 (T_1 - T_\kappa) R^2}{\mu \ln \kappa} \right] \ln \xi$$

which is just Eq. 10B.10-2, along with the definitions of A and B .

c. Integration of Eq. 10B.10.2 then yields

$$v_z = \frac{1}{2} A \xi^2 + \frac{1}{4} B (\xi^2 \ln \xi - \xi^2) + C_1 \ln \xi + C_2$$

The constants of integration are

$$C_2 = -\frac{1}{4}(A - B) \quad \text{and} \quad C_1 = \frac{(A - B)(1 - \kappa^2) - B\kappa^2 \ln \kappa}{4 \ln \kappa}$$

The velocity distribution can then be written as

$$4v_z = (B - A) \left[(1 - \xi^2) - (1 - \kappa^2) \frac{\ln \xi}{\ln \kappa} \right] + B(\xi^2 - \kappa^2) \ln \xi \quad (*)$$

This can easily be shown to satisfy the boundary conditions for the velocity at the inner and outer walls of the annular region.

It remains to evaluate the constant A (as we did in §10.9) by equating the total mass flow through a cross-section to zero:

$$\int_0^{2\pi} \int_{\kappa R}^R (\rho v_z) r dr d\theta = 0 \quad \text{or} \quad \int_{\kappa}^1 [\rho_1 - \rho_1 \beta_1 (T - T_1)] v_z \xi d\xi = 0$$

Thus we have to evaluate A from

$$\int_{\kappa}^1 \left[1 + \beta_1 (T_1 - T_{\kappa}) \frac{\ln \xi}{\ln \kappa} \right] \left\{ (B - A) \left[(1 - \xi^2) - (1 - \kappa^2) \frac{\ln \xi}{\ln \kappa} \right] + B(\xi^2 - \kappa^2) \ln \xi \right\} \xi d\xi = 0$$

First we note that $B \propto \Delta T \equiv T_1 - T_{\kappa}$ and therefore that $A \propto \Delta T$. To be consistent with the Boussinesq approximation, we should therefore neglect terms that are proportional to $(\Delta T)^2$. This eliminates the need for doing many of the integrations in the above equation. When the necessary integrations are performed we get, after a modest amount of algebra

$$(B - A) = \frac{B \ln \kappa}{4} \left[\frac{(1 - \kappa^2)(1 - 3\kappa^2) - \kappa^4 \ln \kappa}{(1 - \kappa^2)^2 + (1 - \kappa^4) \ln \kappa} \right]$$

When this is substituted into (*) along with the expression for B given just after Eq. 10B.10-2, we get Eq. 10B.10-3.

10B.11 Free convection with temperature-dependent viscosity

a. It is instructive and helpful to begin by going back and re-solving the problem of §9.8 using the notation of the present problem. Equations 10.9-9 to 11 in dimensionless form are

$$\frac{d^2\tilde{v}_z}{d\tilde{y}^2} = P + \frac{1}{2}\text{Gr}\tilde{y} \quad \text{with} \quad \tilde{v}_z(\pm 1) = 0$$

This may be integrated to give

$$\tilde{v}_z = C_2 + C_1\tilde{y} + \frac{1}{2}P\tilde{y}^2 + \frac{1}{12}\text{Gr}\tilde{y}^3$$

When the boundary conditions are applied, we get $C_1 = -\frac{1}{12}\text{Gr}$ and $C_2 = -\frac{1}{2}P$. Thus the velocity distribution is

$$\tilde{v}_z = \frac{1}{12}\text{Gr}(\tilde{y}^3 - \tilde{y}) + \frac{1}{2}P(\tilde{y}^2 - 1)$$

which is equivalent to Eq. 10.9-12. Next we apply Eq. 10.9-13, which, in dimensionless form is

$$\int_{-1}^{+1}(1 - b_T\tilde{y})\left[\frac{1}{12}\text{Gr}(\tilde{y}^3 - \tilde{y}) + \frac{1}{2}P(\tilde{y}^2 - 1)\right]d\tilde{y} = 0$$

in which $b_T = \frac{1}{2}\bar{\beta}\Delta T$; this equation has to be solved for P . In doing the integrations, the integrals over odd powers of \tilde{y} give zero, and we get

$$\text{Gr}\left(-\frac{1}{12}\cdot\frac{2}{5} + \frac{1}{12}\cdot\frac{2}{3}\right)b_T + P\left(-\frac{1}{2}\cdot 2 + \frac{1}{2}\cdot\frac{2}{3}\right) = 0 \quad \text{from which} \quad P = \frac{1}{30}\text{Gr}b_T$$

When this value for P is substituted into the dimensionless velocity distribution we get

$$\tilde{v}_z = \frac{1}{12}\text{Gr}(\tilde{y}^3 - \tilde{y}) + \frac{1}{60}\text{Gr}b_T(\tilde{y}^2 - 1)$$

On the right side, the first term contains ΔT and the second term contains $(\Delta T)^2$. In §9.8 we chose to neglect the term in $(\Delta T)^2$, which is

all right if ΔT is quite small. This led us to the result in Eq. 10.9-17, which is just the first term in the above equation.

b. Now we address the temperature-dependent viscosity problem in Problem 10B.11. The equation of motion is then:

$$\frac{d}{dy} \left(\mu(T) \frac{dv_z}{dy} \right) = \frac{dp}{dz} + \rho g$$

We now introduce the temperature dependence of the viscosity (Eq. 10B.11-1) and density (Eq. 10.9-6) to get

$$\frac{d}{dy} \left(\frac{\bar{\mu}}{1 + \bar{\beta}_\mu (T - \bar{T})} \frac{dv_z}{dy} \right) = \frac{dp}{dz} + \bar{\rho} g - \bar{\rho} g \bar{\beta} (T - \bar{T})$$

We now multiply this equation by $B^3 \bar{\rho} / \bar{\mu}^2$ in order to get the equation in dimensionless form (using quantities defined in Eqs. 10B.11-2 and 3):

$$\frac{d}{d\tilde{y}} \left(\frac{1}{1 - b_\mu \tilde{y}} \frac{d\tilde{v}_z}{d\tilde{y}} \right) = P + \frac{1}{2} \text{Gr} \tilde{y} \quad \text{with } \tilde{v}_z(\pm 1) = 0$$

Integration of this equation gives

$$\tilde{v}_z = C_2 + C_1 \tilde{y} + \frac{1}{2} (P - C_1 b_\mu) \tilde{y}^2 + \frac{1}{3} \left(\frac{1}{4} \text{Gr} - P b_\mu \right) \tilde{y}^3 - \frac{1}{16} \text{Gr} b_\mu \tilde{y}^4$$

Application of the boundary conditions then gives

$$C_1 = -\frac{1}{3} \left(\frac{1}{4} \text{Gr} - P b_\mu \right) \quad \text{and} \quad C_2 = -\frac{1}{2} P + \frac{1}{48} \text{Gr} b_\mu + \frac{1}{6} P b_\mu^2$$

In (a) we found that P was second order in ΔT . Therefore the dashed underlined terms in C_1 and C_2 can be expected to be of the third and fourth order respectively--thus two orders of magnitude higher than the remainder of the terms. Therefore we drop the dashed-underlined terms at this point.

Then the dimensionless velocity distribution is

$$\tilde{v}_z = \left(-\frac{1}{2}P + \frac{1}{48}Grb_\mu\right) - \frac{1}{12}Gr\tilde{y} + \frac{1}{2}\left(P + \frac{1}{12}Grb_\mu\right)\tilde{y}^2 + \frac{1}{12}Gr\tilde{y}^3 - \frac{1}{16}Grb_\mu\tilde{y}^4$$

Next we determine P from the requirement that there be no net mass movement upward in the region between the plates (i.e., Eq. 10.9-13):

$$0 = 2\left(-\frac{1}{2}P + \frac{1}{48}Grb_\mu\right) + \frac{2}{3}b_T\left(\frac{1}{12}Gr\right) + \frac{2}{3} \cdot \frac{1}{2}\left(P + \frac{1}{12}Grb_\mu\right) - \frac{2}{5}b_T\left(\frac{1}{12}Gr\right) - \frac{2}{5}\frac{1}{16}Grb_\mu$$

When this is solved for P we get

$$P = \frac{1}{30}Grb_T + \frac{1}{15}Grb_\mu$$

and this is second order in ΔT as we had anticipated, and the dropping of the dashed underlined terms is fully justified. When this expression for P is substituted into the dimensionless velocity distribution we finally obtain:

$$\tilde{v}_z = \frac{1}{12}Gr(\tilde{y}^3 - \tilde{y}) + \frac{1}{60}Grb_T(\tilde{y}^2 - 1) - \frac{1}{80}Grb_\mu(\tilde{y}^2 - 1)(5\tilde{y}^2 - 1)$$

The first term is the "basic solution" in Eq. 10.9-17, the second term is the deviation from the basic solution when terms second order in ΔT are accounted for, and the third term is the second order term that enters in when the temperature dependence of the viscosity is taken into account. The basic solution contains only odd powers of the coordinate, whereas the second order terms contain only even powers. It is clear that this equation satisfies the boundary conditions.

10B.12 Heat conduction with temperature-dependent thermal conductivity

The thermal conductivity is a function of the temperature:

$$k = k_0 - (k_0 - k_\pi) \left(\frac{T_0 - T}{T_0 - T_\pi} \right) \equiv k_0 - (k_0 - k_\pi) \Theta$$

a. Since T is a function of θ alone, we make a shell energy balance thus:

$$q_\theta|_\theta (r_2 - r_1)L - q_\theta|_{\theta+\Delta\theta} (r_2 - r_1)L = 0$$

Dividing by $(r_2 - r_1)L\Delta\theta$ and taking the limit as $\Delta\theta \rightarrow 0$ gives

$$\frac{dq_\theta}{d\theta} = 0$$

Then inserting Fourier's law with a temperature-dependent thermal conductivity, we get

$$\frac{d}{d\theta} \left[-k \frac{1}{r} \frac{dT}{d\theta} \right] = 0 \quad \text{or} \quad \frac{d}{d\theta} \left[-(k_0 - (k_0 - k_\pi)\Theta) \frac{1}{r} \frac{d\Theta}{d\theta} \right] = 0$$

Integrating once we get

$$(k_0 - (k_0 - k_\pi)\Theta) \frac{d\Theta}{d\theta} = C_1$$

and a second integration gives

$$k_0\Theta - \frac{1}{2}(k_0 - k_\pi)\Theta^2 = C_1\theta + C_2$$

When the constants are determined from the boundary conditions, $\Theta(0) = 0$ and $\Theta(\pi) = 1$, we get $C_2 = 0$ and $C_1 = (k_0 + k_\pi)/2\pi$, so that the temperature distribution in the solid is

$$\frac{k_0 \Theta - \frac{1}{2}(k_0 - k_\pi) \Theta^2}{\frac{1}{2}(k_0 + k_\pi)} = \frac{\theta}{\pi}$$

Then the total heat flow through the surface at $\theta = 0$ is then

$$\begin{aligned} Q &= \int_0^L \int_{r_1}^{r_2} \left(-k \frac{1}{r} \frac{dT}{d\theta} \right) \Big|_{\theta=0} dr dz \\ &= k_0 (T_0 - T_\pi) \int_0^L \int_{r_1}^{r_2} \left(\frac{1}{r} \frac{d\Theta}{d\theta} \right) \Big|_{\theta=0} dr dz \\ &= k_0 (T_0 - T_\pi) \int_0^L \int_{r_1}^{r_2} \frac{1}{r} \left(\frac{\frac{1}{2}(k_0 + k_\pi)}{\pi k_0} \right) dr dz \\ &= \left(L \ln \frac{r_2}{r_1} \right) \left(\frac{k_0 + k_\pi}{2} \right) \left(\frac{T_0 - T_\pi}{\pi} \right) \end{aligned}$$

10B.13 Flow reactor with exponentially temperature-dependent source

For this problem we may take T_0 to be zero, since there is no particular need to do otherwise. Then the temperature is made dimensionless by dividing by the inlet temperature: $\Theta = T/T_1$. Also, the quantity S_{c1} may be identified with K of Eq. 10B.13-1. Therefore the quantity $S_{c1}F(\Theta)$ in Eq. 10.5-7 becomes

$$S_{c1}F(\Theta) = K \exp\left(-\frac{E}{RT}\right) = K \exp\left(-\frac{A}{\Theta}\right) \quad \text{where} \quad A = \frac{E}{RT_1}$$

We may then proceed to Eqs. 10.b-21 to 23:

$$\text{Zone I:} \quad \Theta^I = T^I/T_1 = 1 \quad \text{or} \quad T^I = T_1$$

$$\text{Zone II:} \quad \int_{\Theta^I}^{\Theta^{II}} \frac{1}{F(\Theta)} d\Theta = \int_1^{\Theta^{II}} \exp\left(+\frac{A}{\Theta}\right) d\Theta = NZ$$

To do the integral we make a change of variables $A/\Theta = x$ so that $d\Theta = -Ax^{-2}dx$. Then the integral becomes

$$-A \int_1^{A/\Theta^{II}} e^x x^{-2} dx = +A \frac{e^x}{x} \Big|_1^{A/\Theta^{II}} - A \int_1^{A/\Theta^{II}} e^x x^{-1} dx = NZ$$

This last integral can be written as a power series. We then get:

$$A \exp(A/\Theta^{II}) - Ae - A \left[\ln x + \sum_{n=1}^{\infty} \frac{x^n}{n!n} \right] \Big|_1^{A/\Theta^{II}} = NZ \quad \text{or}$$

$$\exp(A/\Theta^{II}) - e - \sum_{n=1}^{\infty} \frac{1}{n!n} + \ln(A/\Theta^{II}) + \sum_{n=1}^{\infty} \frac{(A/\Theta^{II})^n}{n!n} = \frac{NZ}{A}$$

for Θ^{II} as a function of Z . However it is easier to calculate Z as a function of Θ^{II} , since that does not involve any trial and error.

Zone III: $\Theta^{\text{III}} = \Theta^{\text{II}} \Big|_{Z=1}$

Another way to treat Zone II:

If the temperature rise is not too great, we may expand $e^x x^{-2}$ in a power series about $x = 1$:

$$e^x x^{-2} = e \left[1 - (x-1) + \frac{3}{2!}(x-1)^2 - \frac{11}{3!}(x-1)^3 + \frac{33}{4!}(x-1)^4 + \dots \right]$$

Then

$$\begin{aligned} -A \int_1^{A/\Theta^{\text{II}}} e^x x^{-2} dx &= -Ae \int_1^{A/\Theta^{\text{II}}} \left[1 - (1-x) + \frac{3}{2}(1-x)^2 \dots \right] dx \\ &= -Ae \int_0^{(A/\Theta^{\text{II}})-1} \left[1 - y + \frac{3}{2}y^2 \dots \right] dy \\ &= -Ae \left[\left(\frac{A}{\Theta^{\text{II}}} - 1 \right) - \frac{1}{2} \left(\frac{A}{\Theta^{\text{II}}} - 1 \right)^2 + \frac{3}{2} \left(\frac{A}{\Theta^{\text{II}}} - 1 \right)^3 + \dots \right] = NZ \end{aligned}$$

for Θ^{II} as a function of Z . Also, by this method it is easier to calculate Z as a function of Θ^{II} , since that does not involve any trial and error.

10B.14 Evaporation loss from an oxygen tank

a. The fact that the thermal conductivity varies linearly with temperature can be written as

$$\frac{k - k_0}{k_1 - k_0} = \frac{T - T_0}{T_1 - T_0} \equiv \Theta$$

which also serves to define the dimensionless temperature. From an energy balance on a solid spherical shell of thickness Δr gives

$$(4\pi r^2 q_r)|_r - (4\pi r^2 q_r)|_{r+\Delta r} = 0$$

Division by Δr and then taking the limit as Δr goes to zero gives

$$\frac{d}{dr}(r^2 q_r) = 0$$

Then, introduction of Fourier's law gives

$$\frac{d}{dr}\left(r^2 k \frac{dT}{dr}\right) = 0 \quad \text{or} \quad \frac{d}{dr}\left(r^2 k \frac{d\Theta}{dr}\right) = 0$$

We now integrate once with respect to r to get

$$r^2 k \frac{d\Theta}{dr} = C_1 \quad \text{or} \quad k \frac{d\Theta}{dr} = \frac{C_1}{r^2}$$

Inserting the expression for the thermal conductivity as a function of temperature, and integrating again, we find

$$\int [k_0 + (k_1 - k_0)\Theta] d\Theta = -\frac{C_1}{r} + C_2$$

or

$$k_0 \Theta + \frac{1}{2}(k_1 - k_0)\Theta^2 = -\frac{C_1}{r} + C_2$$

Now we apply the boundary conditions:

$$\text{B. C. at } r_0: \quad 0 = -\frac{C_1}{r_0} + C_2$$

$$\text{B. C. at } r_1: \quad \frac{1}{2}(k_1 + k_0) = -\frac{C_1}{r_1} + C_2$$

Subtracting the first equation from the second eliminates C_2 (which we are not going to need anyway) and gives an equation for C_1 :

$$\frac{1}{2}(k_1 + k_0) = C_1 \left(\frac{1}{r_0} - \frac{1}{r_1} \right)$$

or

$$C_1 = \frac{\frac{1}{2}(k_1 + k_0)r_0r_1}{(r_1 - r_0)}$$

The heat flow *inward* through the spherical shell at r_0 is now

$$Q_0 = 4\pi r_0^2 \left(+k \frac{dT}{dr} \right) \Big|_{r=r_0} = 4\pi r_0^2 \left(+k_0 (T_1 - T_0) \frac{d\Theta}{dr} \right) \Big|_{r=r_0}$$

The derivative of the dimensionless temperature can be found thus:

$$k_0 \frac{d\Theta}{dr} + (k_1 - k_0)\Theta \frac{d\Theta}{dr} = \frac{\frac{1}{2}(k_1 + k_0)r_0r_1}{(r_1 - r_0)} \frac{1}{r^2}$$

Evaluating this expression at $r = r_0$ gives

$$k_0 \frac{d\Theta}{dr} \Big|_{r=r_0} = \frac{\frac{1}{2}(k_1 + k_0)r_1}{(r_1 - r_0)r_0}$$

Putting this expression into the formula for Q_0 gives

$$Q_0 = 4\pi r_0^2 \left(+k \frac{dT}{dr} \right) \Big|_{r=r_0} = 4\pi r_0 r_1 \left(\frac{k_0 + k_1}{2} \right) \left(\frac{T_1 - T_0}{r_1 - r_0} \right)$$

b Since most of the quantities are given in the c.g.s. system, we will convert all quantities into that system. The quantity Q_0 may be evaluated thus:

$$Q_0 = 4\pi(36 \times 2.54)(48 \times 2.54) \left(\frac{(16.2 \times 10^{-4})(4.136 \times 10^{-3})}{2} \right) \left(\frac{183}{12 \times 2.54} \right)$$

$$= 2.8182 \text{ cal/s}$$

The factor 4.136×10^{-3} needed for converting the units of the thermal conductivity is obtained from Table F.3-5. The rate of evaporation of oxygen is then:

$$R = \left(\frac{2.8182}{1636} \right) (3600) = 6.201 \text{ g-ml/hr}$$

or

$$R = \frac{(6.201)(32)}{1000} = 0.198 \text{ kg/hr}$$

10B.15 Radial temperature gradients in an annular chemical reactor

a. Consider a cylindrical shell of thickness Δr and length L . We make an energy balance over this shell, by paralleling the derivation in Eqs. 10.2-2 to 6, replacing the electrical heat source by the chemical heat source S_c . Hence we have (cf. Eq. 10.2-6):

$$\frac{d}{dr}(rq_r) = S_c r$$

Into this we substitute Fourier's law for heat conduction in the r -direction to get

$$\frac{d}{dr}\left(r\left(-k_{\text{eff}}\frac{dT}{dr}\right)\right) = S_c r \quad \text{or} \quad k_{\text{eff}}\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -S_c$$

provided that the effective thermal conductivity does not vary with position. The boundary conditions are:

$$\text{B. C. 1:} \quad \text{at } r = r_0, \quad T = T_0$$

$$\text{B. C. 2:} \quad \text{at } r = r_0, \quad \frac{dT}{dr} = 0$$

b. A natural choice for the dimensionless radial coordinate involves division of r by either the inner or outer radius; we choose the inner radius and write $\xi = r/r_0$. Then the differential equation becomes:

$$\frac{k_{\text{eff}}}{r_0^2}\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dT}{d\xi}\right) = -S_c \quad \text{or} \quad \frac{k_{\text{eff}}}{r_0^2}\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d(T-T_0)}{d\xi}\right) = -S_c$$

From this, it is evident that $k_{\text{eff}}(T-T_0)/S_c r_0^2$ is dimensionless. By inserting a factor of 4, we get $\Theta = 4k_{\text{eff}}(T-T_0)/S_c r_0^2$. The insertion of the factor of 4 is arbitrary, but it makes the final dimensionless answer somewhat simpler. In terms of these dimensionless quantities the partial differential equation becomes:

$$\frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{d\Theta}{d\xi} \right) = -4$$

with boundary conditions

B. C. 1: at $\xi = 1$, $\Theta = 0$

B. C. 2: at $\xi = 1$, $d\Theta/d\xi = 0$

c. Integration twice leads to

$$\Theta = -\xi^2 + C_1 \ln \xi + C_2$$

Application of the boundary conditions gives $C_1 = 2$ and $C_2 = 1$, so that

$$\Theta = 1 - \xi^2 + 2 \ln \xi$$

d. The dimensionless temperature at the outer wall is then

$$\Theta(a) = 1 - a^2 + 2 \ln a \quad (a = r_1/r_0)$$

The volume averaged reduced temperature is:

$$\begin{aligned} \langle \Theta \rangle &= \frac{\int_0^{2\pi} \int_1^a (1 - \xi^2 + 2 \ln \xi) \xi d\xi d\theta}{\int_0^{2\pi} \int_1^a \xi d\xi d\theta} = \frac{\left[\frac{1}{2} \xi^2 - \frac{1}{4} \xi^4 + \xi^2 \ln \xi - \frac{1}{2} \xi^2 \right]_1^a}{\left[\frac{1}{2} \xi^2 \right]_1^a} \\ &= -\frac{1}{2} (a^2 + 1) + 2 \frac{a^2 \ln a}{a^2 - 1} \end{aligned}$$

e. The temperature at the outer wall is

$$\begin{aligned} T &= T_0 + \frac{S_c r_0^2}{4k_{\text{eff}}} (1 - a^2 + 2 \ln a) \\ &= 900 + \frac{\left(4800 \frac{\text{cal}}{\text{hr} \cdot \text{cm}^3} \right) \left(3.97 \times 10^{-3} \frac{\text{Btu}}{\text{cal}} \right) \left(2.54 \times 12 \frac{\text{cm}}{\text{ft}} \right) \left(\frac{0.45}{12} \right)^2}{4(0.3)} \end{aligned}$$

$$\begin{aligned} & \times(1 - (1.11)^2 + 2\ln(1.11)) \\ & = 900 + (0.681)(1 - 1.23 + 0.21) \\ & = 900 + (63.3)(1 - 1.23 + 0.21) = 899^\circ\text{F} \end{aligned}$$

f. If the inner and outer radii were doubled, the temperature difference between the walls would be four times as great.

10B.16 Temperature distribution in a hot-wire anemometer

a. We start by writing a shell balance over a segment Δz of the wire:

$$q_z|_z \left(\frac{1}{4} \pi D^2\right) - q_z|_{z+\Delta z} \left(\frac{1}{4} \pi D^2\right) + \frac{I^2}{k_e} \left(\frac{1}{4} \pi D^2\right) \Delta z - h(T - T_L) \pi D \Delta z = 0$$

Division by $\frac{1}{4} \pi D^2 \Delta z$ and taking the limit as $\Delta z \rightarrow 0$ gives

$$-\lim_{\Delta z \rightarrow 0} \frac{q_z|_{z+\Delta z} - q_z|_z}{\Delta z} + \frac{I^2}{k_e} - \frac{4h(T - T_L)}{D} = 0$$

Taking the limit and using the definition of the first derivative gives:

$$-\frac{dq_z}{dz} - \frac{4h(T - T_L)}{D} = -\frac{I^2}{k_e} \quad \text{or} \quad +k \frac{d^2 T}{dz^2} - \frac{4h(T - T_L)}{D} = -\frac{I^2}{k_e}$$

Let us now define the dimensionless quantities

$$\zeta = \frac{z}{L} \quad \Theta = \frac{T - T_L}{T_L} \quad H = \frac{4hL^2}{kD} \quad J = \frac{I^2 L^2}{k_e k T_L}$$

Now when the differential equation is multiplied by L^2/kT_L it becomes, in dimensionless form,

$$\frac{d^2 \Theta}{d\zeta^2} - H\Theta = -J \quad \text{with} \quad \Theta = 0 \text{ at } \zeta = \pm 1$$

This is a nonhomogeneous, second-order differential equation, the solution of which is the sum of a complementary function and a particular integral:

$$\Theta_{\text{C.F.}} = C_1 \cosh \sqrt{H}\zeta + C_2 \sinh \sqrt{H}\zeta; \quad \Theta_{\text{P.I.}} = J/H$$

Since the solution must be symmetric about $\zeta = 0$, the constant C_2 must be zero. Therefore, the complete solution has the form

$$\Theta = C_1 \cosh \sqrt{H} \zeta + \frac{J}{H} \quad \text{where} \quad C_1 = -\frac{J/H}{\cosh \sqrt{H}}$$

where the boundary condition that $\Theta = 0$ at $\zeta = 1$ has been used. Therefore the solution to the problem is

$$\Theta = \frac{J}{H} \left(1 - \frac{\cosh \sqrt{H} \zeta}{\cosh \sqrt{H}} \right) \quad \text{or} \quad T - T_L = \frac{I^2 D}{4hk_e} \left(1 - \frac{\cosh \sqrt{4h/kD} z}{\cosh \sqrt{4h/kD} L} \right)$$

c. First we evaluate the cosh-function in the denominator, creating dimensionless ratios with internally consistent units:

$$\begin{aligned} \cosh \sqrt{\frac{4h}{kD}} L &= \cosh \sqrt{\frac{4hL^2}{kD}} = \cosh \sqrt{\left(\frac{4hL}{k} \right) \left(\frac{L}{D} \right)} = \\ &= \cosh \sqrt{\frac{4(100)((0.5)/(2.54)(12))}{(40.2)} \left(\frac{0.5}{0.0127} \right)} = \cosh 2.535 = 6.348 \end{aligned}$$

Then, at $z = 0$, the quantity in the large parentheses is

$$\left(1 - \frac{1}{\cosh \sqrt{4h/kD} L} \right) = 0.842$$

Then the volume rate of heat production is

$$\begin{aligned} \frac{I^2}{k_e} &= \frac{(T - T_L)(4h)}{(0.842)D} = \frac{(30)(4)(100 \times 5.678)}{(0.842)(0.127 \times 0.001)} = 6.371 \times 10^8 \text{ J/m}^3 \text{ s} \\ &= 6.371 \times 10^2 \text{ J/cm}^3 \text{ s} = 6.371 \times 10^2 \text{ amp}^2 \cdot \text{ohm/cm}^3 \end{aligned}$$

since $1 \text{ amp} \cdot \text{volt} = 1 \text{ amp}^2 \text{ ohm} = 1 \text{ J/s}$. Then

$$\begin{aligned} \text{current (amp)} &= \sqrt{6.371 \times 10^2 \times 10^5 \left(\frac{1}{4} \pi (0.0127)^2 \right)} \\ &= \sqrt{63.71 \times 10^6 \left(1.267 \times 10^{-4} \right)} = 1.01 \text{ amp} \end{aligned}$$

10B.17 Non-Newtonian flow with forced-convection heat transfer

a. This problem can be solved by modifying §10.8 as follows: First replace $(1 - \xi^2)$ by $(1 - \xi^{s+1})$ in Eqs. 10.8-19, 25, and 26. Then the power-law analog of Eq. 10.8-27 becomes

$$\Theta(\xi, \zeta) = C_0 \zeta + C_0 \left(\frac{\xi^2}{4} - \frac{\xi^{s+3}}{(s+3)^2} \right) + C_1 \ln \xi + C_2$$

in which the constants of integration are determined to be

$$C_1 = 0$$

$$C_0 = 2(s+3)/(2+1)$$

$$C_2 = [(s+3^3) - 8]/4(s+1)(s+3)(s+5)$$

With these values for the constants, we are led to Eq. 1B.17-1.

b. This problem may be solved by modifying Problem 10B.7, part (b), by replacing $(1 - \sigma^2)$ by $(1 - \sigma^{s+1})$ in the differential equation for Ψ as well as in the integral condition. This leads to the following equation for $\Theta(\sigma, \zeta)$:

$$\Theta(\sigma, \zeta) = C_0 \zeta + C_0 \left(\frac{\sigma^2}{2} - \frac{\sigma^{s+3}}{(s+2)(s+3)} \right) + C_1 \sigma + C_2$$

with the constants of integration given as

$$C_1 = 0$$

$$C_0 = \frac{s+2}{s+1}$$

$$C_2 = -\frac{s+2}{s+1} \left(\frac{(s+2)(s+3)(2s+5) - 6}{6(s+3)(s+4)(2s+5)} \right)$$

Then we are led to Eq. 10B.17-2.

10B.18 Reactor temperature profiles with axial heat flux

a. The differential equations in Eqs. 10.5-6 to 8, and the boundary conditions in Eqs. 10.5-9 to 14 are still valid, but with a linear form for the function F :

$$S_c = S_{c1} \left(\frac{T - T_0}{T_1 - T_0} \right) \equiv S_{c1} \Theta$$

in which S_{c1} and T_0 are constants describing the linear dependence of the reaction rate on the temperature. Then, if we use the dimensionless downstream coordinate $Z = z/L$, a dimensionless quantity $B = \rho \hat{C}_p v_0 L / k_{\text{eff},zz}$, and a dimensionless chemical heat source $N = S_{c1} L / \rho \hat{C}_p v_0 (T_1 - T_0)$, the differential equations for the three regions become:

$$\frac{1}{B} \frac{d^2 \Theta^I}{dZ^2} = \frac{d\Theta^I}{dZ}; \quad \frac{1}{B} \frac{d^2 \Theta^{II}}{dZ^2} = \frac{d\Theta^{II}}{dZ} + N\Theta^{II}; \quad \frac{1}{B} \frac{d^2 \Theta^{III}}{dZ^2} = \frac{d\Theta^{III}}{dZ}$$

The boundary conditions are:

B.C.1:	at $Z = -\infty$,	$\Theta^I = 1$
B.C.2:	at $Z = 0$,	$\Theta^I = \Theta^{II}$
B.C.3:	at $Z = 0$,	$d\Theta^I/dZ = d\Theta^{II}/dZ$
B.C.4:	at $Z = 1$,	$\Theta^{II} = \Theta^{III}$
B.C.5:	at $Z = 1$,	$d\Theta^{II}/dZ = d\Theta^{III}/dZ$
B.C.6:	at $Z = \infty$,	$\Theta^{III} = \text{finite}$

That is, we specify the condition far upstream from the reaction region, and we require that there be continuity of the temperature and the heat flux where the regions join.

The solutions of these linear, homogeneous, second-order differential equations are then

Region I:	$\Theta^I(Z) = C_1 + C_2 e^{BZ}$
Region II:	$\Theta^{II}(Z) = C_3 e^{m_+ Z} + C_4 e^{m_- Z}$ for $m_+ \neq m_-$
Region III:	$\Theta^{III}(Z) = C_5 + C_6 e^{BZ}$

where $m_{\pm} = \frac{1}{2}B(1 \pm \sqrt{1 - (4N/B)})$.

When the integration constants are determined, we get:

$$\text{Region I: } \Theta^{\text{I}}(Z) = 1 + \left(\frac{m_+ m_- (m_+ - m_-)}{m_+^2 e^{m_+} - m_-^2 e^{m_-}} \right) e^{(m_+ + m_-)Z}$$

$$\text{Region II: } \Theta^{\text{II}}(Z) = \left(\frac{m_+ e^{m_+} e^{m_- Z} - m_- e^{m_-} e^{m_+ Z}}{m_+^2 e^{m_+} - m_-^2 e^{m_-}} \right) (m_+ + m_-)$$

$$\text{Region III: } \Theta^{\text{III}}(Z) = \left(\frac{m_+^2 - m_-^2}{m_+^2 e^{m_+} - m_-^2 e^{m_-}} \right) e^{(m_+ + m_-)Z}$$

These results correspond to Eqs. 10.5-21 to 23.

b. In preparation for taking the limit for B going to infinity, we first find the following Taylor series expansions

$$m_+ = \frac{1}{2}B(1 + (1 - \frac{1}{2}(4N/B) + \dots)) = B + N + O(B^{-1})$$

$$m_- = \frac{1}{2}B(1 - (1 - \frac{1}{2}(4N/B) + \dots)) = N + O(B^{-1})$$

It is important to note that the limiting value of m_- for infinite B is not zero, but N .

Then, the limiting expressions for the temperature profiles in the three regions are

$$\lim_{B \rightarrow \infty} \Theta^{\text{I}}(Z) = \lim_{B \rightarrow \infty} \left[1 + \left(\frac{m_+ m_- e^{m_+} - \dots}{m_+^2 e^{m_+} - \dots} \right) e^{(m_+ + m_-)Z} \right] = 1 \quad (\text{since } Z \text{ is negative})$$

$$\lim_{B \rightarrow \infty} \Theta^{\text{II}}(Z) = \lim_{B \rightarrow \infty} \left(\frac{m_+ (m_+ + \dots) e^{m_+} e^{m_- Z} - m_- (m_+ + \dots) e^{m_-} e^{m_+ Z}}{m_+^2 e^{m_+} - \dots} \right) = e^{NZ}$$

$$\lim_{B \rightarrow \infty} \Theta^{\text{III}}(Z) = \lim_{B \rightarrow \infty} \left(\frac{m_+^2 - \dots}{m_+^2 e^{m_+} - \dots} \right) e^{(m_+ + m_-)Z} = e^N$$

These results are consistent with Eqs. 10.5-21, 22, and 23. To get the second equation above from Eq. 10.5-22, we have to substitute $F(\Theta) = \Theta$ and integrate.

10C.1 Heating of an electric wire with temperature-dependent electrical and thermal conductivity

a. Rewrite Eq. 10C.1-3 as

$$-\frac{k_0 T_0}{R^2} \left[\frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{k}{k_0} \frac{d\Theta}{d\xi} \right) \right] = k_{e0} \left[\frac{k_e}{k_{e0}} \left(\frac{E}{L} \right)^2 \right]$$

This is easily transformed into Eq. 10C.1-4, with $B = k_{e0} R^2 E^2 / k_0 T_0 L^2$. Use of Eqs. 10C.1-1 and 2 leads directly to Eq. 10C.1-5.

b. When Eq. 10C.1-8 is substituted into Eq. 10C.1-5 we get

$$\begin{aligned} & -\frac{1}{\xi} \frac{d}{d\xi} \left(\left[1 - \alpha_1 \left\{ \frac{1}{4} B (1 - \xi^2) (1 + B\Theta_1 + \dots) \right\} \right. \right. \\ & \left. \left. - \alpha_2 \left\{ \frac{1}{4} B (1 - \xi^2) (1 + B\Theta_1 + \dots) \right\}^2 + \dots \right] \xi \frac{d}{d\xi} \left(\frac{1}{4} B (1 - \xi^2) (1 + B\Theta_1 + \dots) \right) \right) \\ & = B \left(1 - \beta_1 \left\{ \frac{1}{4} B (1 - \xi^2) (1 + B\Theta_1 + \dots) \right\} \right. \\ & \quad \left. - \beta_2 \left\{ \frac{1}{4} B (1 - \xi^2) (1 + B\Theta_1 + \dots) \right\}^2 + \dots \right) \end{aligned}$$

Equating terms containing B^1 , we get

$$-\frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{d}{d\xi} \frac{1}{4} B (1 - \xi^2) \right) = B$$

which, when the differentiations are performed, gives an identity, $B = B$, as it should. Then we equate the terms containing B^2 , which gives

$$\begin{aligned} & -\frac{1}{\xi} \frac{d}{d\xi} \left[\xi \frac{d}{d\xi} \frac{1}{4} B (1 - \xi^2) B\Theta_1 \right] - \frac{1}{\xi} \frac{d}{d\xi} \left[-\alpha_1 \frac{1}{4} B \xi \frac{d}{d\xi} \frac{1}{4} B (1 - \xi^2) \right] \\ & = -\beta_1 B \cdot \frac{1}{4} B (1 - \xi^2) \end{aligned}$$

Division by $\frac{1}{4} B^2$ and performing the differentiation in the α_1 term then gives the differential equation for Θ_1

$$\frac{1}{\xi} \frac{d}{d\xi} \left[\xi \frac{d}{d\xi} (1 - \xi^2) \Theta_1 \right] = -\alpha_1 (1 - 2\xi^2) + \beta_1 (1 - \xi^2)$$

Performing two integrations then gives

$$(1 - \xi^2) \Theta_1 = -\frac{1}{8} \alpha_1 (2\xi^2 - \xi^4) + \frac{1}{16} \beta_1 (4\xi^2 - \xi^4) + C_1 \ln \xi + C_2$$

The constant C_1 must be zero in order to satisfy the boundary condition at the tube axis, and the application of the boundary condition at the tube wall gives

$$0 = -\frac{1}{8} \alpha_1 + \frac{3}{16} \beta_1 + C_2$$

Taking the difference between the last two equations gives (after dividing through by $(1 - \xi^2)$)

$$\Theta_1 = \frac{1}{8} \alpha_1 (1 - \xi^2) - \frac{1}{16} \beta_1 (3 - \xi^2)$$

in agreement with Eq. 10C.1-9.

c. Rearranging Eq. 10.C-10 we have

$$\frac{k}{k_0} = \frac{k_e}{k_{e0}} \frac{T}{T_0} = \frac{k_e}{k_{e0}} (1 + \Theta)$$

Into this equation we substitute Eqs. 10C.1-1 and 2, to get

$$\begin{aligned} (1 - \alpha_1 \Theta - \alpha_2 \Theta^2 - \dots) &= (1 - \beta_1 \Theta - \beta_2 \Theta^2 - \dots) (1 + \Theta) \\ &= 1 - \beta_1 \Theta - \beta_2 \Theta^2 - \dots + \Theta - \beta_1 \Theta^2 - \beta_2 \Theta^3 - \dots \\ &= 1 - (\beta_1 - 1) \Theta - (\beta_2 - \beta_1) \Theta^2 - \dots \end{aligned}$$

Equating the coefficients of equal powers of Θ then gives $\alpha_1 = \beta_1 - 1$, $\alpha_2 = \beta_2 - \beta_1$, and so on. Then Eq. 10C.1-12 follows directly:

$$\Theta = \frac{1}{4} B (1 - \xi^2) \left\{ 1 - \frac{1}{16} B [(\beta_1 + 2) + (\beta_1 - 2) \xi^2] + \dots \right\}$$

10C.2 Viscous heating with temperature-dependent viscosity and thermal conductivity

a. The shell momentum and energy balances lead to

$$\frac{d}{dx} \left(\mu \frac{dv_z}{dx} \right) = 0 \quad \text{and} \quad \frac{d}{dx} \left(k \frac{dT}{dx} \right) + \mu \left(\frac{dv_z}{dx} \right)^2 = 0$$

Multiply the first equation by $b^2/\mu_0 v_b$ and the second by $b^2/\mu_0 v_b^2$ to get Eqs. 10C.2-3 and 4.

b. When Eqs. 10C.2-5, 6, 7, and 8 are used in the equation for temperature, and the coefficients of like powers of Br are equated, we get for the terms containing the first power of Br

$$\frac{d^2 \Theta_1}{d\xi^2} + C_{10}^2 = 0$$

This equation has the solution $\Theta_1 = -\frac{1}{2} C_{10}^2 \xi^2 + C_3 \xi + C_4$. The constants of integration are determined from the boundary conditions that $\Theta_1 = 0$ at $\xi = 0, 1$. Thus the following result is obtained:

$$\Theta_1 = \frac{1}{2} C_{10}^2 (\xi - \xi^2)$$

We now turn to the ϕ equation (along with Eq. 10C.2-2) which is

$$\frac{d\phi}{d\xi} = C_1 (1 + \beta_1 \Theta + \beta_2 \Theta^2 + \dots)$$

The expansions in Eqs. 10C.2-6, 7, and 8 then give

$$\frac{d}{d\xi} (\phi_0 + \text{Br} \phi_1 + \dots) = (C_{10} + \text{Br} C_{11} + \text{Br}^2 C_{12} + \dots) \\ (1 + \beta_1 (\text{Br} \Theta_1 + \text{Br}^2 \Theta_2 + \dots) + \dots)$$

Equating the coefficients of the zeroth power of Br, we get

$$\frac{d}{d\xi} \phi_0 = C_{10}$$

Integration then gives $\phi_0 = C_{10}\xi + C_2$. The boundary conditions $\phi_0(0) = 0$ and $\phi_0(1) = 1$ lead to

$$\phi_0 = \xi \quad \text{and hence also} \quad \Theta_1 = \frac{1}{2}(\xi - \xi^2)$$

These are just the results obtained in §10.4.

We next go back to the energy equation and equate the terms containing Br^2 :

$$\frac{d}{d\xi} \left(\alpha_1 \Theta_1 \frac{d\Theta_1}{d\xi} + \frac{d\Theta_2}{d\xi} \right) + 2C_{11} + \beta_1 \Theta_1 = 0$$

Into this we insert the expression for Θ_1 to obtain

$$\frac{d^2\Theta_2}{d\xi^2} = -\frac{\alpha_1}{4}(1 - 6\xi + 6\xi^2) - 2C_{11} - \frac{1}{2}\beta_1(\xi - \xi^2)$$

Integration twice gives

$$\Theta_2 = -\frac{\alpha_1}{4} \left(\frac{1}{2}\xi^2 - \xi^3 + \frac{1}{2}\xi^4 \right) - C_{11}\xi^2 - \frac{1}{2}\beta_1 \left(\frac{1}{6}\xi^3 - \frac{1}{12}\xi^4 \right) + C_3\xi + C_4$$

Then we use the boundary conditions $\Theta_2 = 0$ at $\xi = 0, 1$ leads to

$$\Theta_2 = -\frac{\alpha_1}{4} \left(\frac{1}{2}\xi^2 - \xi^3 + \frac{1}{2}\xi^4 \right) - \frac{1}{24}\beta_1(-\xi + 2\xi^3 - \xi^4) + C_{11}(\xi - \xi^2)$$

The velocity distribution that goes along with this approximation is ϕ_1 . Equating the terms linear in Br gives the differential equation

$$\frac{d\phi_1}{d\xi} = C_{11} + \beta_1 \Theta_1 = C_{11} + \frac{1}{2}\beta_1(\xi - \xi^2)$$

Integration gives

10C.3 Viscous heating in a cone-and-plate viscometer

a. First we make a table of "translation" to get from the plane Couette flow to the cone-and-plate flow:

<u>Plane Couette Flow</u> (Fig. 10.4-2)	<u>Cone-and-Plate Flow</u> (Fig. 2B.11)
$\xi = x/b$	$\bar{\xi} = \psi/\psi_0$
$\phi = v_z/v_b$	$\bar{\phi} = v_\phi/r\Omega$
v_b/b	Ω/ψ_0
$Br = \mu v_b^2/k_0 T_0$	$\bar{Br}(r/R)^2 = (\mu R^2 \Omega^2/k_0 T_0)(r/R)^2$

The torque on the cone-and-plate viscometer is given by the force times the lever arm integrated over the entire plate:

$$T_z = 2\pi \int_0^R \tau_{\theta\phi} \Big|_{\theta=\pi/2} \cdot r \cdot r dr$$

But

$$\tau_{\theta\phi} = -\mu \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) \approx +\mu \frac{1}{r} \frac{\partial v_\phi}{\partial \psi}$$

is a suitable approximation for the cone-plate system, with $\theta \approx \pi/2$. Therefore the torque expression becomes for a fluid at temperature T_0 and viscosity μ_0

$$T_z = 2\pi \int_0^R \mu_0 \frac{\partial v_\phi}{\partial \psi} \Big|_{\psi=0} r dr \quad (*)$$

When there is no viscous heating, the tangential velocity is

$$v_\phi \approx r\Omega(\psi/\psi_0)$$

Combining the last two equations gives, for no viscous heating:

$$T_{z,0} = \frac{2\pi\mu_0\Omega R^3}{3\psi_0} \quad (**)$$

When (*) is divided by (**), we get

$$\frac{T_z}{T_{z,0}} = 3 \int_0^1 \frac{\partial \phi}{\partial \xi} \Big|_{\xi=0} \xi^2 d\xi$$

b. We now consider the system with viscous heating and use the cone-and-plate variables in the correspondence table. This gives

$$\frac{T_z}{T_{z,0}} = 3 \int_0^1 \left(1 - \frac{1}{3} \overline{\text{Br}} \xi^2 \beta_1 + \dots\right) \xi^2 d\xi$$

where we have used the dimensionless velocity expression from part (c) in Problem 10C.2.

When the integration is performed we get

$$\frac{T_z}{T_{z,0}} = 1 - \frac{1}{5} \overline{\text{Br}} \beta_1 + O(\overline{\text{Br}}^2)$$

To get the higher-order terms shown in Eq. 10C.3-1, one would have to go back and get the higher order terms in part (c) in Problem 10C.2.

10D.1 Heat loss from a circular fin

a. A heat balance over a ring of thickness Δr gives

$$(2\pi(2B)rq_r)\Big|_r - (2\pi(2B)rq_r)\Big|_{r+\Delta r} - 2 \cdot 2\pi r \Delta r \cdot h(T - T_a) = 0$$

Divide by $2\pi(2B)\Delta r$ and take the limit as Δr goes to zero to get

$$\lim_{r \rightarrow 0} \frac{(rq_r)\Big|_r - (rq_r)\Big|_{r+\Delta r}}{\Delta r} - \frac{rh}{B}(T - T_a) = 0 \quad \text{or}$$

from which we get the differential equation

$$-\frac{d}{dr}(rq_r) - \frac{rh}{B}(T - T_a) = 0 \quad \text{or} \quad \frac{d}{dr}\left(r \frac{dT}{dr}\right) - \frac{rh}{Bk}(T - T_a) = 0$$

In the second form, Fourier's law has been introduced. Next we rewrite the equation in terms of dimensionless variables: $\xi = r/R_0$ and $\Theta = (T - T_a)/(T_0 - T_a)$:

$$\frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{d\Theta}{d\xi} \right) - \left(\frac{hR_0^2}{Bk} \right) \Theta = 0$$

This equation has the following solution (with $\beta^2 = hR_0^2/Bk$):

$$\Theta(\xi) = C_1 I_0(\beta\xi) + C_2 K_0(\beta\xi)$$

in which I_0 and K_0 are zero-order Bessel functions. The constants are determined by use of the boundary conditions: $\Theta(1) = 1$ and $d\Theta/d\xi|_{\xi=R_1/R_0} = 0$. This leads to

$$1 = C_1 I_0(\beta) + C_2 K_0(\beta) \quad \text{and} \quad 0 = C_1 \beta I_1(\beta R_1/R_0) - C_2 \beta K_1(\beta R_1/R_0)$$

These two equations can be solved simultaneously to give

$$C_1 = \frac{\beta K_1(\beta R_1/R_0)}{I_0(\beta)\beta K_1(\beta R_1/R_0) + \beta I_1(\beta R_1/R_0)K_0(\beta)}$$

$$C_2 = \frac{\beta I_1(\beta R_1/R_0)}{I_0(\beta) \beta K_1(\beta R_1/R_0) + \beta I_1(\beta R_1/R_0) K_0(\beta)}$$

Hence the dimensionless temperature profile is

$$\Theta(\xi) = \frac{I_0(\beta \xi) K_1(\beta R_1/R_0) + I_1(\beta R_1/R_0) K_0(\beta \xi)}{I_0(\beta) K_1(\beta R_1/R_0) + I_1(\beta R_1/R_0) K_0(\beta)}$$

b. The total heat loss is then:

$$\begin{aligned} Q &= 2\pi R_0 (2B) k \left. \frac{dT}{dr} \right|_{r=R_0} = 4\pi B k (T_0 - T_a) \left. \frac{d\Theta}{d\xi} \right|_{\xi=1} \\ &= 4\pi B k \beta (T_0 - T_a) \left[\frac{I_1(\beta R_1/R_0) K_1(\beta) - I_1(\beta) K_1(\beta R_1/R_0)}{I_1(\beta R_1/R_0) K_0(\beta) + I_0(\beta) K_1(\beta R_1/R_0)} \right] \end{aligned}$$

10D.2 Duct flow with constant wall heat flux and arbitrary velocity distribution

a. The analogs of Eqs. 10.8-19, 26 are

$$\phi(\xi) \frac{\partial \Theta}{\partial \zeta} = \frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial \Theta}{\partial \xi} \right) \quad \text{and} \quad \frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{d\Psi}{d\xi} \right) = C_0 \phi(\xi)$$

After multiplication by ξ the second of these can be integrated once to give

$$\xi \frac{d\Psi}{d\xi} = C_0 \int_0^\xi \bar{\xi} \phi(\bar{\xi}) d\bar{\xi} + C_1$$

Then division by ξ and further integration gives Eq. 10D.2-1 (along with the definition in Eq. 10D.2-2). The constant C_1 has to be zero, since the temperature must be finite at the tube axis (B.C. 1). Then from B.C. 2 we get $C_0 = 1/I(1)$ using the equation immediately above. To get the remaining constant, we use the integral condition analogous to Eq. 10.8-24, to get

$$\begin{aligned} \zeta &= \int_0^1 \Theta(\xi, \zeta) \phi(\xi) \xi d\xi \\ &= \int_0^1 \left[C_0 \zeta + C_0 \int_0^\xi \bar{\xi}^{-1} I(\bar{\xi}) d\bar{\xi} + C_2 \right] \phi(\xi) \xi d\xi \\ &= C_0 \zeta I(1) + C_0 \int_0^1 \phi(\xi) \xi \left[\int_0^\xi \bar{\xi}^{-1} I(\bar{\xi}) d\bar{\xi} \right] d\xi + C_2 I(1) \\ &= \zeta + [I(1)]^{-1} \int_0^1 \phi(\xi) \xi \left[\int_0^\xi \bar{\xi}^{-1} I(\bar{\xi}) d\bar{\xi} \right] d\xi + C_2 I(1) \end{aligned}$$

From the last line, Eq. 10D.2-3 follows immediately.

b. The wall and bulk temperatures, in dimensionless form, are given by

$$\Theta_0 = \Theta(\xi = 0, \zeta) \quad \text{and} \quad \Theta_b = \frac{\int_0^1 \phi \Theta \xi d\xi}{\int_0^1 \phi \xi d\xi}$$

Substitution of the result in (a) into these expressions gives

$$\Theta_0 = \frac{1}{I(1)} \zeta + \frac{1}{I(1)} \int_0^1 \frac{I(1)}{\xi} d\xi + C_2$$

$$\Theta_b = \frac{1}{I(1)} \zeta + \frac{1}{[I(1)]^2} \int_0^1 \phi \left[\int_0^\xi \frac{I(\bar{\xi})}{\bar{\xi}} d\bar{\xi} \right] \xi d\xi + C_2$$

Subtracting and then changing the order of integration in the second expression, we obtain

$$\Theta_0 - \Theta_b = \frac{1}{I(1)} \int_0^1 \frac{I(1)}{\xi} d\xi - \frac{1}{[I(1)]^2} \int_0^1 \frac{I(\bar{\xi})}{\bar{\xi}} \left[\int_{\bar{\xi}}^1 \phi(\xi) \xi d\xi \right] d\bar{\xi}$$

The inner integral may now be written as the difference of two integrals, the first going from 0 to 1 and the second from 0 to $\bar{\xi}$. But these two integrals are, respectively, just $I(1)$ and $I(\bar{\xi})$. Hence we get finally

$$\Theta_0 - \Theta_b = \frac{1}{I(1)} \int_0^1 \frac{I(1)}{\xi} d\xi - \frac{1}{I(1)} \int_0^1 \frac{I(1)}{\xi} d\xi + \frac{1}{[I(1)]^2} \int_0^1 \frac{[I(\bar{\xi})]^2}{\bar{\xi}} d\bar{\xi}$$

Thus, the first two terms cancel each other, and the last term gives Eq. 10D.2-4.

c. From the definition of the dimensionless temperature, we get

$$\Theta_0 - \Theta_b = \frac{T_0 - T_b}{q_0 R / k}$$

Taking the reciprocal, and replacing R by $D/2$, we get Eq. 10D.2-5.

d. The quantity $I(1)$ is the ratio of the average to the maximum velocity in the tube.

$$I(1) = \frac{\int_0^1 \phi \xi d\xi}{\int_0^1 \phi d\xi} = \frac{\int_0^1 v_r \xi d\xi}{\int_0^1 v_r d\xi} = \frac{\langle v_r \rangle}{v_{r, \max}}$$

11A.1 Temperature in a friction bearing.

The method given in the solution of Problem 10A.3, based on Eq. 10.4-9, gives the maximum temperature rise in the lubricant as

$$\begin{aligned} T_{\max} - T_1 &= \frac{1}{8} \frac{\mu \Omega^2 R^2}{k} \\ &= \frac{1}{8} \frac{(2.0 \text{ g/cm}\cdot\text{s})(8000\pi/60 \text{ rad/s})^2 (2.54 \text{ cm})^2}{([4.0 \times 10^{-4} \times 4.1840 \times 10^7] \text{ g}\cdot\text{cm/s}^2\cdot\text{K})} = 16.9\text{C} \end{aligned}$$

and the maximum temperature T_{\max} as 217C.

Next, we consider the analysis given in §11.4, which includes the curvature effects. Since T_1 and T_κ are equal, Eq. 11.4-14 is applicable and gives

$$\xi_{\max} = \sqrt{\frac{2 \ln(1/\kappa)}{((1/\kappa)^2 - 1)}}$$

as the location where the temperature T_{\max} occurs. For this problem, $\kappa = 1/1.002$; hence,

$$\xi_{\max} = \sqrt{\frac{2 \ln(1.002)}{((1.002)^2 - 1)}} = 0.999001165$$

which location is essentially in the middle of the gap.

To evaluate T_{\max} , we multiply Eq. 11.4-13 by $(T_1 - T_\kappa)$ to make each term finite; the result (after division by κ^4 in the numerator and denominator) is

$$T - T_\kappa = \frac{\mu \Omega R^2}{k} \cdot \frac{1}{((1/\kappa)^2 - 1)^2} \left[\left(1 - \frac{1}{\xi^2}\right) - \left(1 - \frac{1}{\kappa^2}\right) \frac{\ln \xi}{\ln \kappa} \right]$$

With $\xi = \xi_{\max}$, we obtain the maximum temperature rise in this system as

$$\begin{aligned} T_{\max} - T_\kappa &= \frac{(2.0 \text{ g/cm}\cdot\text{s})(8000\pi/60 \text{ rad/s})^2 (2.54 \text{ cm})^2}{([4.0 \times 10^{-4} \times 4.1840 \times 10^7] \text{ g}\cdot\text{cm/s}^2\cdot\text{K})} \cdot \frac{1}{(1.002^2 - 1)^2} \\ &\quad \cdot [-0.002000667 + 0.002002667] \\ &= 8.438 \times 10^6 \cdot [2.00 \times 10^6] = 16.9\text{C} \end{aligned}$$

in agreement with the previous calculation. This good agreement is attributed to the narrowness of the gap relative to the cylinder radii in this problem.

11A.2 Viscosity variation and velocity gradients in a nonisothermal film.

a. We begin by determining the temperature at which the logarithmic discrepancy, Δ , between the two viscosity representations in Eq. 11.4-18 is largest. The discrepancy is expressed as follows:

$$\begin{aligned}\Delta &= \ln(\text{First } \mu(x) \text{ function}) - \ln(\text{Approximation to first } \mu(x) \text{ function}) \\ &= \left[B \frac{T_0 - T_\delta}{T_0 T} \left(\frac{x}{\delta} \right) \right] - \left[B \frac{T_0 - T_\delta}{T_0 T_\delta} \left(\frac{x}{\delta} \right) \right]\end{aligned}$$

Use of Eq. 11.4-1 to express (x/δ) in terms of temperatures then gives

$$\begin{aligned}\Delta &= \left[\frac{B}{T_0 T} (T_0 - T_\delta) \left(\frac{T - T_0}{T_\delta - T_0} \right) \right] - \left[\frac{B}{T_0 T_\delta} (T_0 - T_\delta) \left(\frac{T - T_0}{T_\delta - T_0} \right) \right] \\ &= -\frac{B}{T_0} \left[\frac{T - T_0}{T} - \frac{T - T_0}{T_\delta} \right]\end{aligned}$$

The T -derivative of the logarithmic discrepancy is

$$\frac{d\Delta}{dT} = -\frac{B}{T_0} \left[\frac{T_0}{T^2} - \frac{1}{T_\delta} \right]$$

Setting this derivative equal to zero, we get

$$\frac{T_0}{T^2} = \frac{1}{T_\delta} \quad \text{or} \quad T = T_d = \sqrt{T_0 T_\delta}$$

as the temperature of maximum discrepancy between the two expressions for $\ln \mu(x)$.

b. For the conditions given,

$$\begin{aligned}\mu(T_0) &= \mu(80^\circ\text{C}) = 0.3548 \times 10^{-2} \text{ g/cm}\cdot\text{s}; \\ \mu(T_\delta) &= \mu(100^\circ\text{C}) = 0.2821 \times 10^{-2} \text{ g/cm}\cdot\text{s}; \\ T_d &= \sqrt{(273.15 + 80)(273.15 + 100)} = 363.01 \text{ K or } 89.86^\circ\text{C}\end{aligned}$$

Eqs. 11.4-14 and 11.4-19 then give the viscosity at T_d as

$$\begin{aligned}\mu(x_d) &= \mu_0 \exp \left[\left(\frac{\mu_\delta}{\mu_0} \right) \left(\frac{T_d - T_0}{T_\delta - T_0} \right) \right] \\ &= (0.3548 \text{ cp}) \exp \left[\left(\ln \left(\frac{0.2821}{0.3548} \right) \right) \left(\frac{89.86 - 80}{100 - 80} \right) \right] \\ &= (0.3548 \text{ cp}) \exp [(-0.2293)(0.493)] = 0.3169 \text{ cp}\end{aligned}$$

Three-point Newton interpolation of $\ln \mu$ in Table 1.1-2 gives $\mu = 0.3151$ at 89.86°C , so the largest relative discrepancy is $\Delta = -0.0057$, or -0.6 percent of $\mu(x)$.

11A.3 Transpiration cooling.

a. In the absence of transpiration, Eq. 11.4-1 is indeterminate, but its limiting form is obtainable by expressing the exponential functions as first-order Taylor expansions in w_r (and thus in R_0):

$$\frac{T - T_1}{T_\kappa - T_1} = \frac{1/r - 1/R}{1/\kappa R - 1/R}$$

This profile, designated as Θ_0 , is tabulated here for the present geometry:

r , microns	100	200	300	400	500
Θ_0	1.000	0.375	0.1666...	0.0625	0

In the presence of transpiration with the given rate $w_r = 1 \times 10^{-5}$ g/s, the constant R_0 in Eq. 11.4-27 is

$$\begin{aligned} R_0 &= \frac{(1 \times 10^{-5} \text{ g/s})(0.25 \text{ cal/g}\cdot\text{C})}{(4\pi)(6.13 \times 10^{-5} \text{ cal/cm}\cdot\text{s}\cdot\text{C})} \\ &= 0.003245 \text{ cm} = 32.45 \text{ microns} \end{aligned}$$

Equation 11.14-27 then gives, with r in microns,

$$\frac{T - T_1}{T_\kappa - T_1} = \frac{(\exp(-32.45/r) - \exp(-32.45/500))}{((\exp(-32.45/200) - \exp(-32.45/500)))}$$

A table of this function, here called Θ_w , follows:

r , microns	100	200	300	400	500
Θ_w	1.000	0.406	0.185	0.070	0.000

c. The ratio of the heat conduction to the inner surface $r = \kappa R$ with the latter transpiration rate to that with $w_r = 0$ is, from Eq. 11.4-32,

$$\begin{aligned} \frac{Q}{Q_0} &= \frac{\phi}{\exp \phi - 1} \\ &= \frac{(R_0(1 - \kappa)/\kappa R)}{\exp(R_0(1 - \kappa)/\kappa R) - 1} \\ &= \frac{(32.45)(0.8)/100}{\exp(32.45)(0.8)/100 - 1} \\ &= \frac{0.2596}{\exp(0.2596) - 1} = 0.876 \end{aligned}$$

Thus, this small rate of transpiration reduces the rate of heat conduction to the inner surface by 12.4 percent.

11A.4 Free-convection heat loss from a vertical surface.

The physical properties for this problem are as follows, evaluated at an average temperature $\bar{T} = (T_0 + T_1)/2 = 110^\circ\text{F} = 43.33^\circ\text{C} = 316.5\text{K}$:

$$\bar{\beta} = 1/\bar{T} = 0.00316\text{K}^{-1}$$

$$\bar{\rho} = pM/R\bar{T} = 0.001154\text{g/cm}^3$$

$$\mu = 1.923 \times 10^{-4}\text{g/cm}\cdot\text{s from Table 1.1-2}$$

$$\hat{C}_p = 1.007\text{ J/g}\cdot\text{K from Perry's Handbook, 6th Ed., Table 3-212}$$

$$= 0.2407\text{ cal/g}\cdot\text{K}$$

$$k = 0.0276\text{ W/m}\cdot\text{K from Perry's Handbook, 6th Ed., Table 3-212}$$

$$= 6.60 \times 10^{-5}\text{ cal/s}\cdot\text{cm}\cdot\text{K}$$

The Prandtl and Grashof numbers are then

$$\begin{aligned}\text{Pr} &= \frac{\hat{C}_p\mu}{k} \\ &= \frac{(0.2407\text{ cal/g}\cdot\text{K})(1.923 \times 10^{-4}\text{ g/cm}\cdot\text{s})}{6.60 \times 10^{-5}\text{ cal/s}\cdot\text{cm}\cdot\text{K}} = 0.701, \text{ and}\end{aligned}$$

$$\begin{aligned}\text{Gr} &= \frac{\bar{\rho}^2 g \bar{\beta} (T_0 - T_1) H^2}{\mu^2} \\ &= \frac{(0.001154\text{ g/cm}^3)^2 (980.7\text{ cm/s}^2) (0.00316\text{ K}^{-1}) [(150 - 70)/1.8]\text{K} (30\text{ cm})^3}{(1.923 \times 10^{-4}\text{ g/cm}\cdot\text{s})^2} \\ &= 1.34 \times 10^8\end{aligned}$$

The heat loss rate from one side of the plate, according to Eq. 11.4-51, is

$$Q = WHq_{\text{avg}} = WC \cdot k((T_0 - T_1)(\text{GrPr})^{1/4})$$

With Lorenz' value of C , this gives

$$\begin{aligned}Q &= (50\text{ cm})(0.548)(6.60 \times 10^{-5}\text{ cal/s}\cdot\text{cm}\cdot\text{s})([80/1.8]\text{k})(1.34 \times 10^8 \times 0.701)^{1/4} \\ &= 7.9\text{ cal/s}\end{aligned}$$

With the value $C = 0.518$ recommended by Whitaker for air, the result is

$$Q = 7.5\text{ cal/s}$$

However, this value of C is based on a Prandtl number of 0.73 for air, rather than the value 0.701 found at the present conditions from more recent data. Linear extrapolation of the table on page 349 gives a C value of 0.516 at $\text{Pr} = 0.701$, and a corrected heat loss rate of 7.4 cal/s.

11A.5 Velocity, temperature and pressure changes in a shock wave.

a. From Eq. 11.4-7, the initial air velocity is

$$\begin{aligned}v_1 &= \text{Ma}_1 \sqrt{\gamma RT_1/M} \\&= (2) \sqrt{1.4(530 \text{ R})(4.9686 \times 10^4 \text{ lb}_m \text{ft}^2/\text{s}^2 \cdot \text{lb-mol} \cdot \text{R})(28.97 \text{ lb}_m/\text{lb-mol})} \\&= 2256 \text{ ft/s}\end{aligned}$$

b. The final velocity v_∞ is found from Eq. 11.4-75 with $\xi \rightarrow \infty$, so that

$$\begin{aligned}\phi \rightarrow \alpha &= \frac{\gamma - 1}{\gamma + 1} + \frac{2}{\gamma + 1} \frac{1}{\text{Ma}_1^2} \\&= \frac{0.4}{2.4} + \frac{2}{2.4} \frac{1}{4} = 0.375\end{aligned}$$

Then Eq. 11.4-69 gives

$$v_\infty = \phi v_1 = (0.375)(2256) = 846 \text{ ft/s}$$

The final temperature is obtained from the energy balance, Eq. 11.4-65:

$$\begin{aligned}T_\infty &= T_1 + \frac{1}{2} \frac{(v_1^2 - v_\infty^2)}{\hat{C}_p} \\&= (530 \text{ R}) + \frac{1}{2} \frac{(2256^2 - 846^2 \text{ ft}^2/\text{s}^2)/2}{(0.242 \text{ Btu}/\text{lb}_m \cdot \text{R} \times 2.5036 \times 10^4 \text{ lb}_m \text{ft}^2/\text{s}^2 \cdot \text{Btu})} \\&= 891^\circ \text{R}\end{aligned}$$

The final pressure is calculated from the foregoing results by use Eq. 11.4-61 and the ideal gas law:

$$\begin{aligned}p_\infty v_\infty / T_\infty &= p_1 v_1 / T_1; \quad \text{hence} \\p_\infty &= p_1 \left(\frac{v_1}{v_\infty} \right) \left(\frac{T_\infty}{T_1} \right) \\&= (1 \text{ atm}) \left(\frac{2256}{846} \right) \left(\frac{891}{530} \right) \\&= 4.48 \text{ atm}\end{aligned}$$

c. The change in specific internal energy is

$$\begin{aligned}\hat{U} &= \hat{C}_v \Delta T = (\hat{C}_p/\gamma) \Delta T \\&= ([0.242/1.4] \text{ Btu}/\text{lb}_m \cdot \text{R})(891 - 530 \text{ R}) \\&= 62.4 \text{ Btu}/\text{lb}_m,\end{aligned}$$

and the change in specific kinetic energy is

$$\begin{aligned}\Delta \hat{K} &= \frac{1}{2} \frac{(846^2 - 2256^2 \text{ ft}^2/\text{s}^2)}{2.5036 \times 10^4 \text{ lb}_m \text{ft}^2/\text{s}^2 \cdot \text{Btu})} \\&= -87.4 \text{ Btu}/\text{lb}_m\end{aligned}$$

11A.6 Adiabatic frictionless compression of an ideal gas.

The states encountered in such a compression satisfy Eq. 11.4-57,

$$p\rho^{-\gamma} = C_1$$

as well as

$$\frac{p}{\rho T} = R/M$$

Combining these relations, we get

$$\rho T \rho^{-\gamma} = T \rho^{1-\gamma} = C_3$$

Hence, the initial and final states in such a compression satisfy

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1} \right)^{\gamma-1}$$

and the final temperature in the case considered here is

$$T_2 = (460 + 100)(10)^{1.4-1} = 1407^\circ\text{R} = 947^\circ\text{F}$$

11A.7 Effect of free convection on the insulating value of a horizontal air space.

The relevant properties of air at 1 atm and 100°C = 373.15 K are:

$$\mu = 0.02173 \text{ cp from Table 1.1-2} = 2.173 \times 10^{-4} \text{ g/cm}\cdot\text{s}$$

$$\rho = pM/RT = 1/(82.0578 \times 373.15) = 0.0009461 \text{ g/cm}^3$$

$$\beta = 1/T = 0.002680 \text{ K}^{-1}$$

$$\hat{C}_p = 1.015 \text{ J/g}\cdot\text{K from CRC Handbook 2000-2001, pp. 6-1, 6-2}$$

$$k = 31.40 \text{ mW/m}\cdot\text{K from CRC Handbook 2000-2001, p. 6-185}$$

$$= 31.40 \times 10^{-5} \text{ W/cm}\cdot\text{K}$$

$$\text{Pr} = \hat{C}_p \mu / k = \frac{(1.015 \text{ J/g}\cdot\text{K})(2.173 \times 10^{-4} \text{ g/cm}\cdot\text{s})}{31.40 \times 10^{-5} \text{ W/cm}\cdot\text{K}} = 0.703$$

The no-flow state will be stable as long as the Rayleigh number, GrPr , does not exceed its critical value of 1708, given in Ref. 4 of §11.5. This gives the following restriction on the temperature difference:

$$\frac{\rho^2 g \beta (T_1 - T_0) h^3}{\mu^2} \frac{\hat{C}_p \mu}{k} \leq 1708$$

or

$$\begin{aligned} (T_1 - T_0) &\leq \frac{1708 \mu^2}{\rho^2 g \beta h^3 \text{Pr}} \\ &= \frac{1708 (2.173 \times 10^{-4})^2}{(0.0009461 \text{ g/cm}^3)^2 (980.665 \text{ cm/s}^2) (0.002680 \text{ K}^{-1}) (2.5 \text{ cm})^3 (0.703)} \\ &= 3.1 \text{ }^\circ\text{C} \end{aligned}$$

If a very thin metal sheet is placed midway between the two plates, forming two cylindrical chambers of height $h/2$, then a corresponding calculation for the total temperature difference across the two chambers gives the stability condition

$$(T_1 - T_0) \leq (2)(8)(3.1) = 49.6^\circ\text{C}$$

for absence of free convection.

11B.1 Adiabatic frictionless processes in an ideal gas

a. For adiabatic frictionless processes, the fluxes may be set equal to zero. Then the energy equation becomes

$$\rho \hat{C}_p \frac{DT}{Dt} = - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{Dp}{Dt} \quad \text{or} \quad \frac{\rho \tilde{C}_p}{M} \frac{DT}{Dt} = - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{Dp}{Dt}$$

For an ideal gas $\rho = pM/RT$, so that $(\partial \ln \rho / \partial \ln T)_p = -1$. Then

$$\frac{p}{RT} (a + bT + cT^2) \frac{DT}{Dt} = \frac{Dp}{Dt}$$

Hence for a given element of moving fluid

$$\left(\frac{a}{T} + b + cT \right) dT = R \frac{dp}{p}$$

Integration from the initial state ρ_1, T_1 gives answer (a) or

$$a \ln \frac{T}{T_1} + b(T - T_1) + \frac{c}{2}(T^2 - T_1^2) = R \ln \frac{p}{p_1}$$

c. For the data given, we have

$$a \ln \frac{T_2}{T_1} = 3.204 \ln \frac{800}{300} = 3.156 \text{ cal / g - mole} \cdot \text{K}$$

$$b(T_2 - T_1) = 18.41 \times 10^{-3} (800 - 300) = 9.205 \text{ cal / g - mole} \cdot \text{K}$$

$$\frac{c}{2}(T_2^2 - T_1^2) = -\frac{4.48 \times 10^{-6}}{2} (800^2 - 300^2) = -1.231 \text{ cal / g - mole} \cdot \text{K}$$

Summing these results we get

$$R \ln \frac{p_2}{p_1} = 11.124 \quad \text{or} \quad \frac{p_2}{p_1} = \exp \frac{11.124}{1.987} = 270$$

Hence $p_2 = 270 \text{ atm}$

11B.2 Viscous heating in laminar tube flow (asymptotic solutions)

a. From the energy equation we have

$$\rho \hat{C}_p v_z \frac{\partial T}{\partial z} = k \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \mu \left(\frac{\partial v_z}{\partial r} \right)^2$$

Into this we substitute the expression for the velocity distribution of a Newtonian fluid in a circular tube: $v_z = v_{z,\max} \left[1 - (r/R)^2 \right]$. This leads, then, to Eq. 11B.2-1.

b. Integration gives (for the isothermal wall) at large z :

$$T = -\frac{\mu v_{z,\max}^2}{4kR^4} r^4 + C_1 \ln r + C_2$$

The constants may be evaluated using the boundary conditions given in the problem, and the result is given in Eq. 11B.2-2.

c. To get the z dependence of the temperature we perform an average over the cross-section

$$\rho \hat{C}_p v_{z,\max} \int_0^1 (1 - \xi^2) \xi d\xi \cdot \frac{dT}{dz} = \frac{4\mu v_{z,\max}^2}{R^2} \int_0^1 \xi^2 \xi d\xi$$

which leads to the result

$$T - T_1 = \left(4\mu v_{z,\max} / \rho \hat{C}_p R^2 \right) z$$

We then make the postulate in Eq. 11B.2-4, and substitute this postulate into Eq. 11B.2-1 to get the following equation for $f(r)$

$$\rho \hat{C}_p v_{z,\max} (1 - \xi^2) \cdot \frac{4\mu v_{z,\max}}{\rho \hat{C}_p R^2} = \frac{k}{R^2} \frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{df}{d\xi} \right) + \frac{4\mu v_{z,\max}^2}{R^2} \xi^2$$

This equation may be integrated to give

$$f(r) = \left(\mu v_{z,\max}^2 / k \right) \left(\xi^2 - \frac{1}{2} \xi^4 \right)$$

which leads to Eq. 11B.2-5.

11B.3 Velocity distribution in a nonisothermal film

a. When $x = \delta$, the two terms inside the large bracket are equal, but with opposite signs, and hence $v_z = 0$.

b. The second term in the bracket, being a constant, will not contribute to the derivative. The factors in front of the bracket are also constant. Therefore the derivative of the bracket is:

$$\frac{d}{dx}[\dots] = \frac{0 + (1/\delta)\ln(\mu_\delta/\mu_0)}{(\mu_\delta/\mu_0)^{x/\delta}} - \frac{1 - (x/\delta)\ln(\mu_\delta/\mu_0)}{[(\mu_\delta/\mu_0)^{x/\delta}]^2} \left(\frac{\mu_\delta}{\mu_0}\right)^{x/\delta} \ln\left(\frac{\mu_\delta}{\mu_0}\right) \frac{1}{\delta}$$

When we set x equal to zero in this expression we get

$$\left. \frac{d}{dx}[\dots] \right|_{x=0} = (1/\delta)\ln(\mu_\delta/\mu_0) - (1/\delta)\ln(\mu_\delta/\mu_0) = 0$$

Hence at $x = 0$, we have found that $dv_z/dx = 0$

c. Let $M = \ln(\mu_\delta/\mu_0)$ and $X = x/\delta$. Then Eq. 11.4-20 becomes

$$\begin{aligned} v_z &= \frac{\rho g \delta^2 \cos \beta}{\mu_0} \left(\frac{1}{M^2} \right) \left(\frac{1 + MX}{e^{MX}} - \frac{1 + M}{e^M} \right) \\ &= \frac{\rho g \delta^2 \cos \beta}{\mu_0} \left(\frac{1}{M^2} \right) \left(\frac{(1 + MX)e^M - (1 + M)e^{MX}}{e^{M(X+1)}} \right) \\ &= \frac{\rho g \delta^2 \cos \beta}{\mu_0} \left(\frac{(1 + MX)(1 + M + \frac{1}{2}M^2 + \dots) - (1 + M)(1 + MX + \frac{1}{2}M^2X^2 + \dots)}{M^2 e^{M(X+1)}} \right) \\ &= \frac{\rho g \cos \beta}{2\mu_0} \left(\frac{1 - X^2 + O(M)}{e^{M(X+1)}} \right) \end{aligned}$$

When $\mu_\delta \rightarrow \mu_0$ (or $M \rightarrow 0$), the above result simplifies to

$$v_z \rightarrow \frac{\rho g \cos \beta}{2\mu_0} (1 - X^2) = \frac{\rho g \cos \beta}{2\mu_0} \left(1 - \left(\frac{x}{\delta} \right)^2 \right)$$

in agreement with Eq. 2.2-18.

11B.4 Heat conduction in a spherical shell

In this problem, the heat flow is in the θ direction. Then Eq. B.9-3 simplifies to

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dT}{d\theta} \right) = 0$$

The first integration leads to

$$\sin \theta \frac{dT}{d\theta} = C_1 \quad \text{or} \quad \frac{dT}{d\theta} = \frac{C_1}{\sin \theta}$$

The second integration gives

$$T = C_1 \ln \left| \tan \frac{1}{2} \theta \right| + C_2 = C_1 \ln \tan \frac{1}{2} \theta + C_2$$

Since the argument of the tangent function is always less than a right angle, the absolute value sign is not needed.

The constants of integration are obtainable from the boundary conditions, which give:

$$T_1 = C_1 \ln \tan \frac{1}{2} \theta_1 + C_2; \quad T_2 = C_1 \ln \tan \frac{1}{2} (\pi - \theta_1) + C_2$$

We next form the following differences:

$$T_1 - T_2 = C_1 \left[\ln \tan \frac{1}{2} \theta_1 - \ln \tan \frac{1}{2} (\pi - \theta_1) \right] = C_1 \ln \frac{\tan \frac{1}{2} \theta_1}{\tan \frac{1}{2} (\pi - \theta_1)}$$

$$T - T_2 = C_1 \left[\ln \tan \frac{1}{2} \theta - \ln \tan \frac{1}{2} (\pi - \theta_1) \right] = C_1 \ln \frac{\tan \frac{1}{2} \theta}{\tan \frac{1}{2} (\pi - \theta_1)}$$

Finally we get for the temperature distribution in the shell:

$$\frac{T - T_2}{T_1 - T_2} = \frac{\ln \left[\tan \frac{1}{2} \theta / \tan \frac{1}{2} (\pi - \theta_1) \right]}{\ln \left[\tan \frac{1}{2} \theta_1 / \tan \frac{1}{2} (\pi - \theta_1) \right]}$$

This solution clearly satisfies the two boundary conditions.

11B.5 Axial heat conduction in a wire

a. This problem involves purely axial flow of heat (by conduction and convection) so that the energy equation is

$$\rho \hat{C}_p v_z \frac{dT}{dz} = k \frac{d^2T}{dz^2} \quad \text{or} \quad -\frac{\rho \hat{C}_p v}{k} \frac{dT}{dz} = \frac{d^2T}{dz^2} \quad \text{or} \quad -A \frac{dT}{dz} = \frac{d^2T}{dz^2}$$

in which $v_z = -v$, and $A \equiv \rho \hat{C}_p v / k$.

Integration of the differential equation gives

$$-AT = \frac{dT}{dz} + C_1$$

At $z = \infty$, we know that $T = T_\infty$ and $dT/dz = 0$; hence $C_1 = -AT_\infty$. Hence the first-order differential equation becomes

$$\frac{d\Theta}{dz} = -A\Theta \quad \text{where} \quad \Theta \equiv (T - T_\infty)/(T_0 - T_\infty)$$

in which (since $\Theta(0) = 1$)

$$\ln \Theta = -Az + \ln C_2 \quad \text{or} \quad \Theta \equiv \frac{T - T_\infty}{T_0 - T_\infty} = e^{-Az}$$

This is just Eq. 11B.5-1.

b. For temperature-dependent physical properties we have the following energy equation:

$$-\rho \hat{C}_{p\infty} L(\Theta) v \frac{d\Theta}{dz} = \frac{d}{dz} \left(k_\infty K(\Theta) \frac{d\Theta}{dz} \right) \quad \text{or} \quad -A_\infty L(\Theta) \frac{d\Theta}{dz} = \frac{d}{dz} \left(K(\Theta) \frac{d\Theta}{dz} \right)$$

in which $A_\infty = \rho \hat{C}_{p\infty} v / k_\infty$. The first integration gives

$$+A_\infty \int_z^\infty L(\Theta) \frac{d\Theta}{dz} dz = K(\Theta) \frac{d\Theta}{dz} + C_1$$

We now use the boundary conditions that at $z = \infty$, $\Theta = 0$ and $d\Theta/dz = 0$, to find that $C_1 = 0$. The above result may then be written as

$$+A_\infty \int_0^\Theta L(\bar{\Theta}) d\bar{\Theta} = K(\Theta) \frac{d\Theta}{dz} \quad \text{or} \quad -A_\infty \int_0^\Theta L(\bar{\Theta}) d\bar{\Theta} = K(\Theta) \frac{d\Theta}{dz}$$

This equation may be integrated to give

$$-A_\infty \int_0^z dz = \int_1^\Theta \frac{K(\bar{\Theta}) d\bar{\Theta}}{\int_0^{\bar{\Theta}} L(\bar{\Theta}) d\bar{\Theta}}$$

This result simplifies to that in (a) when K and L are both equal to

$z = \infty$.

c. To show that the last equation in (b) satisfies the

11B.6 Transpiration cooling in a planar system

We start with Eq. J of Table 11.4-1, and assume steady-state, negligible change in pressure with distance, and negligible viscous dissipation. Then for constant thermal conductivity this equation simplifies to

$$\rho \hat{C}_p v_y \frac{dT}{dy} = k \frac{d^2T}{dy^2} \quad \text{or} \quad \phi \frac{d\Theta}{d\eta} = \frac{d^2\Theta}{d\eta^2}$$

in which $\Theta = (T - T_L)/(T_0 - T_L)$, $\eta = y/L$, and $\phi = \rho \hat{C}_p v_y L/k$, a constant. This equation is to be solved with the boundary conditions that $\Theta(0) = 1$ and $\Theta(1) = 0$.

Set $p = d\Theta/d\eta$ to get the first-order separable equation

$$\phi p = \frac{dp}{d\eta} \quad \text{or} \quad \frac{dp}{p} = \phi d\eta$$

which may be integrated to give

$$\ln p = \phi \eta + \ln C_1 \quad \text{or} \quad \frac{d\Theta}{d\eta} \equiv p = C_1 e^{\phi \eta}$$

This is also a first-order separable equation and integrating it gives

$$\Theta = C_1 \int e^{\phi \eta} d\eta + C_2 = \frac{C_1}{\phi} e^{\phi \eta} + C_2$$

The constants of integration are then found from the boundary conditions, and we get finally

$$\Theta = \frac{e^{\phi \eta} - e^{\phi}}{1 - e^{\phi}}$$

The heat flux at $y = 0$ is then

$$q_0 = -k \left. \frac{dT}{dy} \right|_{y=0} = -\frac{k(T_0 - T_L)}{L} \left. \frac{d\Theta}{d\eta} \right|_{\eta=0} = \frac{k(T_L - T_0)}{L} \frac{\phi}{1 - e^{\phi}}$$

11B.7 Reduction of evaporation losses by transpiration

a. Without transpiration, we have from Eq. 11.4-31

$$Q_0 = \frac{4\pi\kappa R(T_1 - T_\kappa)}{1 - \kappa} = \frac{4\pi\left(\frac{1}{2}\text{ft}\right)(0.02\text{Btu/hr}\cdot\text{ft}\cdot^\circ\text{F})(327^\circ\text{F})}{\frac{1}{2}}$$
$$= 82\text{Btu/hr}$$

b. With transpiration, we get from Eq. 11.4-30

$$Q = \frac{4\pi R_0 k(T_1 - T_\kappa)}{e^{(R_0/\kappa R)(1-\kappa)} - 1} \quad \text{with } R_0 = w_r \hat{C}_p / 4\pi k$$

We do not know R_0 , but we can get it by using an energy balance in the form

$$Q = w_r \Delta \hat{H}_{\text{vap}} = \left(\frac{4\pi k R_0}{\hat{C}_p} \right) \Delta \hat{H}_{\text{vap}}$$

When this is inserted into the left side of Eq. 11.4-30, we get

$$\frac{\Delta \hat{H}_{\text{vap}}}{\hat{C}_p} = \frac{(T_1 - T_\kappa)}{e^{(R_0/\kappa R)(1-\kappa)} - 1}$$

This may be solved for R_0 , and then the energy balance may be used to convert the result into an expression for Q

$$Q = \frac{4\pi k \Delta \hat{H}_{\text{vap}}}{\hat{C}_p} \left(\frac{\kappa R}{1 - \kappa} \right) \ln \left(\frac{\hat{C}_p (T_1 - T_\kappa)}{\Delta \hat{H}_{\text{vap}}} + 1 \right)$$
$$= \frac{4\pi(0.02)(91.7)}{0.22} \left(\frac{0.5}{0.5} \right) \ln \left(\frac{(0.22)(327)}{(91.7)} + 1 \right)$$
$$= (104.76) \ln(1.7845) = (104.76)(0.5792) = 61 \text{ Btu/hr}$$

11B.8 Temperature distribution in an embedded sphere

a. In both regions the partial differential equation is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) = 0$$

b. At the surface of the embedded sphere the boundary conditions are that $T_1 = T_2$ and that $k_1(\partial T_1/\partial r) = k_0(\partial T_0/\partial r)$.

c. For the temperature field inside the sphere, substitution of Eq. 11B.8-1 into the terms in the above differential equation gives:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) = \left[\frac{3k_0}{k_1 + 2k_0} \right] 2Ar^{-1} \cos \theta$$

$$\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) = - \left[\frac{3k_0}{k_1 + 2k_0} \right] 2Ar^{-1} \cos \theta$$

Outside the sphere we get for the same two terms

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) = 2Ar^{-1} \cos \theta - 2 \left[\frac{k_1 - k_0}{k_1 + 2k_0} \right] \left(\frac{R}{r} \right)^3 Ar^{-1} \cos \theta$$

$$\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) = - \left[1 - \left[\frac{k_1 - k_0}{k_1 + 2k_0} \right] \left(\frac{R}{r} \right)^3 \right] 2Ar^{-1} \cos \theta$$

d. The boundary conditions are also satisfied:

$$\left[\frac{3k_0}{k_1 + 2k_0} \right] AR \cos \theta = \left[1 - \left[\frac{k_1 - k_0}{k_1 + 2k_0} \right] \right] AR \cos \theta$$

$$k_1 \left[\frac{3k_0}{k_1 + 2k_0} \right] A \cos \theta = k_0 \left[\frac{3k_0}{k_1 + 2k_0} \right] A \cos \theta + k_0 \left[3 \frac{k_1 - k_0}{k_1 + 2k_0} \right] A \cos \theta$$

11B.9 Heat flow in a solid bounded by two conical surfaces

a, The temperature has to satisfy the differential equation

$$\frac{d}{d\theta} \left(\sin \theta \frac{dT}{d\theta} \right) = 0$$

b. Two successive integrations give

$$\frac{dT}{d\theta} = \frac{C_1}{\sin \theta} \quad \text{and} \quad T = C_1 \ln \left| \tan \frac{1}{2} \theta \right| + C_2$$

Since in this problem, θ may go from 0 to $\frac{1}{2}\pi$, and hence the tangent will not have negative values. Therefore absolute-value signs are not needed.

c. The integration constants are determined from the following simultaneous equations:

$$T_1 = C_1 \ln \left(\tan \frac{1}{2} \theta_1 \right) + C_2 \quad \text{and} \quad T_2 = C_1 \ln \left(\tan \frac{1}{2} \theta_2 \right) + C_2$$

On solving them, one gets

$$C_1 = \frac{T_1 - T_2}{\ln \tan \frac{1}{2} \theta_1 - \ln \tan \frac{1}{2} \theta_2} = \frac{T_1 - T_2}{\ln \left(\tan \frac{1}{2} \theta_1 / \tan \frac{1}{2} \theta_2 \right)}$$
$$C_2 = \frac{T_2 \ln \tan \frac{1}{2} \theta_1 - T_1 \ln \tan \frac{1}{2} \theta_2}{\ln \tan \frac{1}{2} \theta_1 - \ln \tan \frac{1}{2} \theta_2}$$

d. The θ component of the heat-flux vector is obtained from the first equation in (b) above

$$q_\theta = -k \frac{1}{r} \frac{dT}{d\theta} = -\frac{C_1 k}{r \sin \theta} = -\frac{k}{r \sin \theta} \frac{T_1 - T_2}{\ln \left(\tan \frac{1}{2} \theta_1 / \tan \frac{1}{2} \theta_2 \right)}$$

e. The total heat flow across the conical surface is then

$$Q = \int_0^R \int_0^{2\pi} q_\theta \Big|_{\theta=\theta_1} r \sin \theta_1 d\phi dr = 2\pi R k \frac{T_1 - T_2}{\ln \left(\tan \frac{1}{2} \theta_2 / \tan \frac{1}{2} \theta_1 \right)}$$

11B.10 Freezing of a spherical drop

a. The heat conduction equation for the solid phase is

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dT}{dr} \right) = 0 \quad (R_f \leq r \leq R)$$

Two integrations lead to $T = -(C_1/r) + C_2$. The constants of integration are determined from the boundary conditions

$$\text{B. C. 1: At } r = R_f, T = T_0; \quad \text{B. C. 2: At } r = R, -k \frac{dT}{dr} = h(T - T_\infty)$$

This leads to the following expressions:

$$C_1 = \frac{T_0 - T_\infty}{(1/R) - (1/R_f) - (k/hR^2)}; \quad C_2 = T_0 + \frac{T_0 - T_\infty}{\left[(1/R) - (1/R_f) - (k/hR^2) \right] R_f}$$

The total heat flow across the spherical surface at $r = R$ is then

$$Q = 4\pi R^2 \left(-k \frac{dT}{dr} \right) \Big|_{r=R} = 4\pi R^2 \left(-\frac{k}{R^2} \right) \frac{T_0 - T_\infty}{\left[(1/R) - (1/R_f) - (k/hR^2) \right]}$$

This can be rearranged to give the solution in the text.

b. We now have to equate the heat liberated on freezing at $r = R_f$ to the heat flowing out across the surface at $r = R$:

$$-(\rho \Delta \hat{H}_f) (4\pi R_f^2) \frac{dR_f}{dt} = \frac{h \cdot 4\pi R^2 \cdot (T_0 - T_\infty)}{1 - (hR/k) + (hR^2/kR_f)}$$

Integration then yields

$$-(\rho \Delta \hat{H}_f) \int_R^0 \left[1 - (hR/k) + (hR^2/kR_f) \right] R_f^2 dR_f = hR^2 (T_0 - T_\infty) \int_0^{t_f} dt$$

where t_f is the time for the freezing of the entire droplet. Evaluation of the integrals then leads to the expression in the text.

11B.11 Temperature rise in a catalyst pellet

a. We make an energy balance over a spherical shell of thickness Δr :

$$4\pi r^2 q_r|_r - 4\pi(r + \Delta r)^2 q_r|_{r+\Delta r} + 4\pi r^2 \Delta r S_c = 0$$

Then division by $4\pi\Delta r$ gives

$$\frac{(r^2 q_r)|_{r+\Delta r} - (r^2 q_r)|_r}{\Delta r} - r^2 S_c = 0$$

When the limit is taken that $\Delta r \rightarrow 0$ and use is made of the definition of the first derivative, we get

$$\frac{d}{dr}(r^2 q_r) - r^2 S_c = 0$$

Insertion of Fourier's law then gives

$$\frac{d}{dr}\left(r^2 k \frac{dT}{dr}\right) + r^2 S_c = 0 \quad (**) \quad \text{or} \quad k \frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + r^2 S_c = 0$$

for the appropriate equation describing the heat conduction with heat generation by chemical reaction and constant k .

b. From Eq. B.9=3, with the time-derivative term set equal to zero, and all velocities set equal to zero, and all derivatives other than r derivatives set equal to zero gives the heat conduction equation in spherical coordinates for a system with no chemical reaction. Therefore, we have to add a term describing the heat production per unit volume:

$$k \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + S_c = 0$$

which is the same as the result obtained in (a).

c. The above differential equation may be integrated in a sequence of steps as follows:

$$\frac{d}{dr} \left(r^2 \frac{dT}{dr} \right) = -\frac{S_c r^2}{k}; \quad \left(r^2 \frac{dT}{dr} \right) = -\frac{S_c r^3}{3k} + C_1; \quad \frac{dT}{dr} = -\frac{S_c r}{3k} + \frac{C_1}{r^2}$$

$$T = -\frac{S_c r^2}{6k} - \frac{C_1}{r} + C_2 \quad (***)$$

The constant C_1 must be zero, because neither the temperature nor its gradient are expected to be infinite. The heat loss to the surroundings provides the second boundary condition needed for getting C_2 :

Definition of heat transfer coefficient: $q_r|_{r=R} = h(T_R - T_g)$

From Fourier's law: $q_r|_{r=R} = -k \frac{dT}{dr} \Big|_{r=R} = +\frac{S_c R}{3}$

Equating these expressions: $h(T_R - T_g) = \frac{S_c R}{3}$

Inserting T_R from (**): $h \left(-\frac{S_c R^2}{6k} + C_2 - T_g \right) = \frac{S_c R}{3}$

Solving for C_2 : $C_2 = \frac{S_c R^2}{6k} + \frac{S_c R}{3h} + T_g$

Thus we finally get the temperature profile within the catalyst pellet:

$$T - T_g = \frac{S_c R^2}{6k} \left[1 - \left(\frac{r}{R} \right)^2 \right] + \frac{S_c R}{3h}$$

d..When the heat transfer coefficient goes to infinity, the last term in the temperature distribution drops out.

e. The maximum temperature in the system is

$$T_{\max} - T_g = \frac{S_c R^2}{6k} + \frac{S_c R}{3h} = \frac{S_c R^2}{6k} \left(1 + \frac{2k}{Rh} \right)$$

f. In Eq. (**) one would have to leave k inside the differential operator, and insert the specific r dependence of both k and S_c .

11B.12 Stability of an exothermic reaction system

a. For the postulated steady-state solution

$$k \frac{d^2 T}{dx^2} + S_{c0} \exp(A(T - T_0)) = 0 \quad \text{with} \quad T = T_0 \text{ at } x = \pm B$$

b. Using the given dimensionless variables, we may rewrite the problem as

$$\frac{d^2 \Theta}{d\xi^2} + \lambda e^\Theta = 0 \quad \text{with} \quad \Theta = 0 \text{ at } \xi = \pm 1$$

c. Multiply the differential equation by $2d\Theta/d\xi$ to get

$$2 \frac{d\Theta}{d\xi} \frac{d^2 \Theta}{d\xi^2} + 2\lambda \frac{d\Theta}{d\xi} e^\Theta = 0 \quad \text{or} \quad \frac{d}{d\xi} \left(\frac{d\Theta}{d\xi} \right)^2 + 2\lambda \frac{de^\Theta}{d\xi} = 0$$

Integration then gives

$$\left(\frac{d\Theta}{d\xi} \right)^2 + 2\lambda e^\Theta = C_1$$

We now use the fact that, from the symmetry of the problem, at $\xi = 0$ we must have $d\Theta/d\xi = 0$. Then if we let $\Theta = \Theta_0$ at $\xi = 0$, we can get an expression for C_1 and then write

$$\left(\frac{d\Theta}{d\xi} \right)^2 - 2\lambda (\exp \Theta_0 - \exp \Theta) = 0$$

Note that we have not "evaluated" the integration constant C_1 , we have merely replaced it by Θ_0 , which has a recognizable physical meaning.

c. We next take the square root of both sides:

$$\frac{d\Theta}{d\xi} = \pm \sqrt{2\lambda} \sqrt{\exp \Theta_0 - \exp \Theta}$$

The minus sign has been selected, since the left side of the differential equation is inherently negative, and the quantity under the square-root sign is positive. Then we integrate over half the plate

$$\int_0^{\Theta_0} \frac{d\Theta}{\sqrt{\exp \Theta_0 - \exp \Theta}} = -\sqrt{2\lambda} \int_1^0 d\xi$$

d. The integral can be done analytically by making the change of variable $y^2 = \exp(\Theta - \Theta_0)$:

$$\begin{aligned} \frac{1}{\sqrt{\exp \Theta_0}} \int_0^{\Theta_0} \frac{d\Theta}{\sqrt{1 - \exp(\Theta - \Theta_0)}} \\ &= \exp\left(-\frac{1}{2}\Theta_0\right) \int_{\exp(-\frac{1}{2}\Theta_0)}^1 \frac{2dy}{y\sqrt{1-y^2}} \\ &= 2\exp\left(-\frac{1}{2}\Theta_0\right) \operatorname{arccosh}\left(\exp\frac{1}{2}\Theta_0\right) \end{aligned}$$

The integral over y can be found in an integral table.

Combining these last two results we get

$$\exp\left(-\frac{1}{2}\Theta_0\right) \operatorname{arccosh}\left(\exp\frac{1}{2}\Theta_0\right) = \sqrt{\frac{1}{2}\lambda}$$

e. For $\lambda > 0.88$ no value of Θ_0 can be found. This means that when S_{c0} is too large, or B is too large, or k is too small, then the heat cannot be dissipated fast enough.

This is an important example, because it illustrates that it is not always possible to get a steady-state solution to a physical problem. To do a complete analysis of this problem, it would be necessary to solve the problem with the time-derivative term included.

11B.13 Laminar annular flow with constant wall heat flux

The equation analogous to Eq. 10.8-12 for the annular flow is

$$\rho \hat{C}_p v_{z,\max} \left[1 - \left(\frac{r}{R} \right)^2 - \frac{1 - \kappa^2}{\ln(1/\kappa)} \ln \left(\frac{R}{r} \right) \right] \frac{\partial T}{\partial z} = k \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right)$$

where Eq. 2.4-14 has been used. We now introduce dimensionless variables: $\xi = r/R$, $\zeta = kz/\rho \hat{C}_p v_{z,\max} R^2$, and $\Theta = k(T - T_1)/q_0 R$. Then the above energy equation becomes

$$\left[(1 - \xi^2) - (1 - \kappa^2) \frac{\ln \xi}{\ln \kappa} \right] \frac{\partial \Theta}{\partial \zeta} = \frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial \Theta}{\partial \xi} \right)$$

This is to be solved with the boundary conditions

$$\text{At } r = \kappa R, \quad -k \frac{\partial T}{\partial r} = q_0 \quad \text{or} \quad \text{at } \xi = \kappa, \quad -\frac{\partial \Theta}{\partial \xi} = 1$$

$$\text{At } r = R, \quad -k \frac{\partial T}{\partial r} = 0 \quad \text{or} \quad \text{at } \xi = 1, \quad -\frac{\partial \Theta}{\partial \xi} = 0$$

$$\text{At } z = 0, \quad T = T_1 \quad \text{or} \quad \text{at } \zeta = 0, \quad \Theta = 0$$

We seek an asymptotic solution for large downstream distances of the form $\Theta(\xi, \zeta) = C_0 \zeta + \Psi(\xi)$. The function $\Psi(\xi)$ has to satisfy

$$\frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial \Psi}{\partial \xi} \right) = C_0 \left[(1 - \xi^2) - (1 - \kappa^2) \frac{\ln \xi}{\ln \kappa} \right]$$

The first and second integrations lead to (cf. Eq. 10.8-27)

$$\frac{\partial \Psi}{\partial \xi} = C_0 \left[\left(\frac{\xi}{2} - \frac{\xi^3}{4} \right) - \frac{1 - \kappa^2}{\ln \kappa} \left(\frac{\xi}{2} \ln \xi - \frac{\xi}{4} \right) \right] + \frac{C_1}{\xi}$$

$$\Psi = C_0 \left[\left(\frac{\xi^2}{4} - \frac{\xi^4}{16} \right) - \frac{1 - \kappa^2}{\ln \kappa} \left(\frac{\xi^2}{4} \ln \xi - \frac{\xi^2}{4} \right) \right] + C_1 \ln \xi + C_2$$

The constants C_0 and C_1 are determined from the boundary conditions at $\xi = \kappa$ and $\xi = 1$:

$$C_0 = \frac{4\kappa}{(1-\kappa^4) + \frac{(1-\kappa^2)^2}{\ln \kappa}} \quad \text{and} \quad C_1 = -\frac{\kappa \left(1 + \frac{1-\kappa^2}{\ln \kappa}\right)}{(1-\kappa^4) + \frac{(1-\kappa^2)^2}{\ln \kappa}}$$

Then the equation for the radial distribution of the temperature is

$$\Psi = \frac{\kappa \xi^2 \left[\left(1 - \frac{\xi^2}{4}\right) - \frac{1-\kappa^2}{\ln \kappa} (\ln \xi - 1) \right] - \kappa \left(1 + \frac{1-\kappa^2}{\ln \kappa}\right) \ln \xi}{(1-\kappa^4) + \frac{(1-\kappa^2)^2}{\ln \kappa}} + C_2$$

The constant C_2 can be obtained by using an integral condition, similar to that in Eq. 10.8-24 or 25

$$2\pi\kappa Rz \cdot q_0 = \int_0^{2\pi} \int_{\kappa R}^R \rho \hat{C}_p (T - T_1) v_z r dr d\theta \quad \text{or} \quad \zeta = \frac{1}{\kappa} \int_{\kappa}^1 \Theta \phi \xi d\xi$$

where $\phi = v_z / v_{z,\max}$. This gives

$$\zeta = \frac{1}{\kappa} \int_{\kappa}^1 \left\{ C_0 \zeta + C_0 \left[\left(\frac{\xi^2}{4} - \frac{\xi^4}{16} \right) - \frac{1-\kappa^2}{\ln \kappa} \left(\frac{\xi^2}{4} \ln \xi - \frac{\xi^2}{4} \right) \right] + C_1 \ln \xi + C_2 \right\} \left[(1-\xi^2) - (1-\kappa^2) \frac{\ln \xi}{\ln \kappa} \right] \xi d\xi$$

or

$$0 = \frac{1}{\kappa} \int_{\kappa}^1 \left\{ C_0 \left[\left(\frac{\xi^2}{4} - \frac{\xi^4}{16} \right) - \frac{1-\kappa^2}{\ln \kappa} \left(\frac{\xi^2}{4} \ln \xi - \frac{\xi^2}{4} \right) \right] + C_1 \ln \xi + C_2 \right\} \left[(1-\xi^2) - (1-\kappa^2) \frac{\ln \xi}{\ln \kappa} \right] \xi d\xi$$

since the terms containing ζ just cancel.

This last equation has to be solved for C_2 . This is in principle an easy problem, but extremely time-consuming and unrewarding. It can, however, be solved by using Mathematica, and the result is*

$$C_2 = \frac{\kappa \left[-9(1-\kappa^2)^3(11+3\kappa^2) - 2(1-\kappa^2)^2(88+88\kappa^2+25\kappa^4)\ln\kappa \right. \\ \left. + 3(-25-24\kappa^2+18\kappa^4+24\kappa^6+7\kappa^8)(\ln\kappa)^2 - 72\kappa^4(\ln\kappa)^3 \right]}{72(1-\kappa^2)^2((1-\kappa^2)+(1+\kappa^2)(\ln\kappa))^2}$$

 *The authors wish to thank Mr. Richard M. Jendrejack for his help on this problem.

11B.14 Unsteady-state heating of a sphere

a. The time-dependent heat conduction equation is

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) \quad \text{or} \quad \frac{\partial \Theta}{\partial \tau} = \frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \Theta}{\partial \xi} \right)$$

In the second form, we have introduced the dimensionless variables:

$\Theta = (T - T_0)/(T_1 - T_0)$, $\xi = r/R$, and $\tau = \alpha t/R^2$. The solution in Eq. 11B.14-1 is, in terms of the same dimensionless variables

$$\Theta = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \frac{\sin n\pi\xi}{n\pi\xi} e^{-n^2\pi^2\tau}$$

b. We do the differentiations; then it is clear that the first and last expression below are the same. Therefore Eq. 11B.14-1 satisfies the differential equation.

$$\frac{\partial \Theta}{\partial \tau} = 2 \sum_{n=1}^{\infty} (-1)^n (-n^2\pi^2) \frac{\sin n\pi\xi}{n\pi\xi} e^{-n^2\pi^2\tau}$$

$$\xi^2 \frac{\partial \Theta}{\partial \xi} = 2 \sum_{n=1}^{\infty} (-1)^n n\pi \left[\frac{\xi \cos n\pi\xi}{n\pi} - \frac{\sin n\pi\xi}{(n\pi)^2} \right] e^{-n^2\pi^2\tau}$$

$$\frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \Theta}{\partial \xi} \right) = 2 \sum_{n=1}^{\infty} (-1)^n n\pi \left[\frac{\cos n\pi\xi}{n\pi} - \frac{n\pi\xi \sin n\pi\xi}{n\pi} - \frac{\cos n\pi\xi}{n\pi} \right] e^{-n^2\pi^2\tau}$$

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \Theta}{\partial \xi} \right) = 2 \sum_{n=1}^{\infty} (-1)^n n\pi \left[-\frac{n\pi \sin n\pi\xi}{n\pi\xi} \right] e^{-n^2\pi^2\tau}$$

c. When $\xi = 1$, we get from Eq. 11B.14-1

$$\Theta = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \frac{\sin n\pi}{n\pi} e^{-n^2\pi^2\tau} = 1$$

since $\sin n\pi = 0$ for all integral values of n .

d. Inasmuch as

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

the solution for $\xi = 0$ is certainly finite:

$$\Theta = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2 \pi^2 \tau}$$

e. In dimensionless form Eq. 11B.14-2 is

$$-1 = 2 \sum_{n=1}^{\infty} (-1)^n \frac{\sin n\pi\xi}{n\pi\xi}$$

Then, multiplying by $\xi \sin m\pi\xi$ and integrating gives

$$-\int_0^1 \xi \sin m\pi\xi d\xi = 2 \sum_{n=1}^{\infty} (-1)^n \frac{1}{n\pi} \int_0^1 \sin m\pi\xi \sin n\pi\xi d\xi$$

The left side can be evaluated as follows:

$$\begin{aligned} -\int_0^1 \xi \sin m\pi\xi d\xi &= -\frac{1}{(m\pi)^2} \int_0^{m\pi} x \sin x dx = -\frac{1}{(m\pi)^2} (\sin x - x \cos x) \Big|_0^{m\pi} \\ &= +\frac{1}{(m\pi)^2} (m\pi \cos m\pi) = \frac{1}{m\pi} \cos m\pi = \frac{1}{m\pi} (-1)^m \end{aligned}$$

The right side may be evaluated thus:

$$\begin{aligned} 2 \sum_{n=1}^{\infty} (-1)^n \frac{1}{n\pi} \int_0^1 \sin n\pi\xi \sin m\pi\xi d\xi &= 2 \sum_{n=1}^{\infty} (-1)^n \frac{1}{n\pi^2} \int_0^{\pi} \sin nx \sin mx dx \\ &= 2 \sum_{n=1}^{\infty} (-1)^n \frac{1}{n\pi^2} \left(\frac{\pi}{2} \delta_{mn} \right) = \frac{1}{m\pi} (-1)^m \end{aligned}$$

Thus the two sides are equal, and it is proven that the initial condition is indeed satisfied.

11B.15 Dimensionless variables for free convection

a. When the proposed dimensionless quantities are introduced, Eqs. 11B.15-1 to 3 follow directly. Nothing more needs to be said.

b. We can convert Eq. 11B.15-1 into Eq. 11.4-44 if we require that

$$(I) \quad \frac{v_{z0}y_0}{v_{y0}H} = 1$$

Then, when (I) is substituted into Eq. 11B.15.2, then the latter may be converted into Eq. 11.4-45 if we also require that

$$(II) \quad \frac{\mu}{\rho y_0 v_{y0}} = \text{Pr}$$

$$(III) \quad \frac{y_0 g \beta (T_0 - T_1)}{v_{y0} v_{z0}} = \text{Pr}$$

Next we substitute (I) into Eq. 11B.15-3 and further require that

$$(IV) \quad \frac{k}{\rho \hat{C}_p y_0 v_{y0}} = 1$$

then Eq. 11B.15-3 becomes Eq. 11.4-35.

We thus have four equations from which to determine the three "scale factors" y_0 , v_{y0} , and v_{z0} . However, it can be seen that Eqs. (II) and (IV) are not independent, since multiplication of Eq. (IV) by Pr just gives Eq. (II). Thus we have three independent equations from which to determine three unknowns.

We now eliminate y_0 by multiplying Eqs. (I) and (II) to get Eq. (V), and by dividing Eq. (III) by Eq. (I) to get Eq. (VI):

$$(V) \quad \frac{v_{z0}\mu}{v_{y0}^2 H \rho} = \text{Pr}$$

$$(VI) \quad \frac{g\beta(T_0 - T_1)H}{v_{z0}^2} = \text{Pr}$$

Introducing the abbreviation $B = \rho g\beta(T_0 - T_1)$, Eq. (VI) can be solved for v_{z0} to give Eq. (VII), and then v_{y0} is obtained from Eq. (V) to give Eq. (VIII):

$$(VII) \quad v_{z0} = \sqrt{\frac{BH}{\rho \text{Pr}}} = \sqrt{\frac{\alpha BH}{\mu}}$$

$$(VIII) \quad v_{y0} = \sqrt{\frac{\mu}{H\rho \text{Pr}}} \sqrt{\frac{\alpha BH}{\mu}} = \sqrt[4]{\frac{\alpha^3 B}{\mu H}}$$

Finally y_0 may be obtained from Eqs. (IV) and (VIII):

$$(IX) \quad y_0 = \frac{k}{\rho \hat{C}_p v_{y0}} = \alpha \sqrt[4]{\frac{\mu H}{\alpha^3 B}} = \sqrt[4]{\frac{\alpha \mu H}{B}}$$

The reciprocals of these last three quantities appear in Eqs. 11.B-41 to 43.

c. If all the dimensionless groups were set equal to unit, then combination of Eqs. (II) and (IV) would give $\text{Pr} = 1$, thereby severely restricting the applicability of the results.

11C.1 The speed of propagation of sound waves

a. Equation 11C.1-1 can also be written as

$$\left(\frac{\partial p}{\partial V}\right)_S = \frac{(\partial H/\partial T)_p}{(\partial U/\partial T)_V} \left(\frac{\partial p}{\partial V}\right)_T \quad \text{or} \quad \frac{(\partial p/\partial S)_V}{(\partial V/\partial S)_p} = \frac{(\partial H/\partial T)_p}{(\partial U/\partial T)_V} \frac{(\partial p/\partial T)_V}{(\partial V/\partial T)_p}$$

This may be rearranged to give

$$\left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial U}{\partial T}\right)_V = \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial V}{\partial S}\right)_p \left(\frac{\partial H}{\partial T}\right)_p$$

Next we apply the Maxwell relations to the first two factors on the left and right sides, and we apply two of the four "fundamental relations" for pure fluids to modify the third factor

$$\left[-\left(\frac{\partial T}{\partial V}\right)_S\right] \left[-\left(\frac{\partial S}{\partial p}\right)_T\right] \left[T\left(\frac{\partial S}{\partial T}\right)_V\right] = \left[\left(\frac{\partial S}{\partial V}\right)_T\right] \left[\left(\frac{\partial T}{\partial p}\right)_S\right] \left[T\left(\frac{\partial S}{\partial T}\right)_p\right]$$

Then we apply the relation

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

to the first and third factors, to obtain

$$\frac{(\partial S/\partial p)_T}{(\partial V/\partial S)_T} = \frac{(\partial S/\partial V)_T}{(\partial p/\partial S)_T}$$

On cross-multiplications, this gives an identity.

b. The equations of continuity and motion are

$$\frac{\partial}{\partial t}(\rho_0 + \rho') = -(\nabla \cdot (\rho_0 + \rho')(\mathbf{v}_0 + \mathbf{v}')); \quad \frac{\partial}{\partial t}(\rho_0 + \rho')(\mathbf{v}_0 + \mathbf{v}') = -\nabla(p_0 + p')$$

Since ρ_0 and p_0 are constants and \mathbf{v}_0 is zero, we get (when the products of the primed quantities are neglected)

$$\frac{\partial \rho}{\partial t} = -\rho_0(\nabla \cdot \mathbf{v}) \quad \text{and} \quad \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p$$

c. Since the momentum and energy fluxes have been taken to be zero), the flow is isentropic. This enables us to rewrite the equation of motion thus:

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -(\nabla \rho) \left(\frac{\partial p}{\partial \rho} \right)_s = -(\nabla \rho) \gamma \left(\frac{\partial p}{\partial \rho} \right)_T$$

To get the second expression, Eq. 11C.1-1 has been used. Use of the definition of the speed of sound then gives Eq. 11C.1-4.

d. Next we take the time derivative of the equation of continuity to get

$$\frac{\partial^2 \rho}{\partial t^2} = -\frac{\partial}{\partial t} [\rho_0(\nabla \cdot \mathbf{v})] = -\left(\nabla \cdot \rho_0 \frac{\partial \mathbf{v}}{\partial t} \right) \quad \text{or} \quad \frac{\partial^2 \rho}{\partial t^2} = v_s^2 \nabla^2 \rho$$

To get the second form, the equation of motion of Eq. 11C.1-4 has been used.

e. From Eq. 11C.1-6 we get

$$\frac{\partial^2 \rho}{\partial t^2} = -\rho_0 A \left(\frac{2\pi}{\lambda} \right)^2 v_s^2 \sin \left[\frac{2\pi}{\lambda} (z - v_s t) \right]$$

$$v_s^2 \frac{\partial^2 \rho}{\partial z^2} = -v_s^2 \rho_0 A \left(\frac{2\pi}{\lambda} \right)^2 \sin \left[\frac{2\pi}{\lambda} (z - v_s t) \right]$$

Therefore, Eq. 11C.1-6 is satisfied.

11C.2 Free convection in a slot

a. The equations of change are simplified as follows:

Continuity:

$$\frac{\partial}{\partial t} \rho = - \left(\frac{\partial}{\partial x} \rho v_x \right) \quad \text{and at steady state and } v_x = 0 \text{ we get } 0 = 0$$

Motion:

$$\frac{\partial}{\partial t} \rho v_z = - \left(\frac{\partial}{\partial z} \rho v_z v_z \right) - \frac{\partial p}{\partial z} - \bar{\rho} g + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} \right) + \bar{\rho} \bar{\beta} g (T - \bar{T}) \quad \text{or}$$

$$0 = \mu \frac{\partial^2 v_z}{\partial x^2} + \bar{\rho} \bar{\beta} g (T - \bar{T}) \quad \text{since} \quad p \approx -\bar{\rho} g z + \text{constant}$$

Energy:

$$0 = k \frac{d^2 T}{dy^2} \quad \text{whence} \quad T = \bar{T} + Ay$$

b. The boundary conditions are;

$$v_z(\pm B, y) = 0 \quad v_z \text{ is an odd function of } y \quad T(x, 0) - \bar{T} = 0.$$

c. To get the velocity profiles we rearrange the equation of motion as follows:

$$\frac{\partial^2 v_z}{\partial x^2} = - \frac{\bar{\rho} \bar{\beta} g A}{\mu} y$$

This may be integrated to give:

$$v_z(x, y) = - \frac{\bar{\rho} \bar{\beta} g A}{2\mu} x^2 y + C_1 x + C_2 \quad \text{or} \quad v_z(x, y) = \frac{\bar{\rho} \bar{\beta} g A B^2}{2\mu} \left(1 - \left(\frac{x}{B} \right)^2 \right) y$$

d. For water at 20°C, $\bar{\rho} = 0.9982 \text{ g/cc}$ and $\bar{\beta} = 0.00021 \text{ (}^\circ\text{C)}^{-1}$.
The maximum velocity will occur at $x = 0$ and $y = W$, so that

$$v_{z,\max} = \frac{\bar{\rho}\bar{\beta}gAB^2W}{2\mu}$$

Hence the corresponding temperature gradient will be

$$\begin{aligned}
 A &= \frac{2\mu v_{z,\max}}{\bar{\rho}\bar{\beta}gB^2W} \\
 &= \frac{2(1.0019 \times 10^{-2} \text{ g/cm} \cdot \text{s}) \left(\frac{(0.001)(2) \text{ cm}}{3600 \text{ s}} \right)}{(0.99823 \text{ g/cm}^3)(0.00021 \text{ }^\circ\text{C}^{-1})(980.7 \text{ cm/s}^2)(0.01 \text{ cm})^2(0.2 \text{ cm})} \\
 &= 0.271 \text{ }^\circ\text{C/cm}
 \end{aligned}$$

11C.3 Tangential annular flow of a highly viscous liquid

First, rewrite Eq. 11.4-13 to eliminate N in favor of the Brinkman number Br , which appears in Eq. 10.4-9:

$$\Theta = \left(1 - \frac{\ln \xi}{\ln \kappa}\right) + Br \frac{\kappa^4}{(1 - \kappa^2)^2} \left[\left(1 - \frac{1}{\xi^2}\right) - \left(1 - \frac{1}{\kappa^2}\right) \frac{\ln \xi}{\ln \kappa} \right]$$

To show that this reduces to Eq. 10.4-9 in the limit of a very thin annulus, begin by taking the limit of the term that does not contain Br and then we treat the term containing Br .

Term without Br :

We let $\kappa = 1 - \varepsilon$ and $\xi = 1 - \varepsilon(1 - \eta)$, where η is the x/b of §10.4. Then using Eq. C.3-2 we find

$$1 - \frac{\ln \xi}{\ln \kappa} = 1 - \frac{-[\varepsilon(1 - \eta) + \frac{1}{2}\varepsilon^2(1 - \eta)^2 + \dots]}{-[\varepsilon + \frac{1}{2}\varepsilon^2 + \dots]} = 1 - (1 - \eta) + O(\varepsilon) = \eta + O(\varepsilon)$$

In the limit of vanishing ε , this leads to the last term in Eq. 10.4-9.

Term with Br :

The coefficient of Br is now

$$\frac{(1 - \varepsilon)^4}{(2\varepsilon - \varepsilon^2)^2} \left[\left(1 - \frac{1}{[1 - \varepsilon(1 - \eta)]^2}\right) - \left(1 - \frac{1}{(1 - \varepsilon)^2}\right) \frac{\varepsilon(1 - \eta) + \frac{1}{2}\varepsilon^2(1 - \eta)^2 + \dots}{\varepsilon + \frac{1}{2}\varepsilon^2 + \dots} \right]$$

When everything is expanded in terms of powers of ε , we find

$$\frac{1}{4\varepsilon^2} \frac{1 - 4\varepsilon + 6\varepsilon^2 + \dots}{1 - \varepsilon + \frac{1}{4}\varepsilon^2 + \dots} \left[\left(1 - [1 + 2\varepsilon(1 - \eta) + 3\varepsilon^2(1 - \eta)^2 + \dots]\right) - \left(1 - [1 + 2\varepsilon + 3\varepsilon^2 + \dots]\right) \left([1 - \eta] + \frac{1}{2}\varepsilon(-\eta + \eta^2) + \dots\right) \right]$$

Simplification of this expression leads to

$$\frac{1}{4\varepsilon^2} \left(1 - 3\varepsilon + \frac{11}{4}\varepsilon^2 + \dots\right) \left[\left(-2\varepsilon(1-\eta) - 3\varepsilon^2(1-\eta)^2\right) \right. \\ \left. - \left(-2\varepsilon - 3\varepsilon^2 - \dots\right) \left([1-\eta] + \frac{1}{2}\varepsilon(-\eta + \eta^2) + \dots\right) \right]$$

Then cancellation of some terms gives

$$\frac{1}{4\varepsilon^2} (1 - 3\varepsilon + \dots) \left[-3\varepsilon^2(1-\eta)^2 - \left\{ 3\varepsilon^2(1-\eta) - \varepsilon^2(-\eta + \eta^2) + \dots \right\} \right] \\ = \frac{1}{4} (1 - 3\varepsilon + \dots) \left[-3(-\eta + \eta^2) + (-\eta + \eta^2) + \dots \right] \\ = \frac{1}{2} (\eta - \eta^2)$$

This is in agreement with the term in Eq. 10.4-9 that is multiplied by Br.

11C.4 Heat conduction with variable thermal conductivity

First we obtain an expression for the gradient of F :

$$\begin{aligned}\nabla F &= \nabla \left(\int k dT + \text{constant} \right) \\ &= \int (\nabla k) dT \\ &= \int \frac{dk}{dT} \nabla T dT\end{aligned}$$

The ∇ operator can be taken inside the integral, since it involves only differentiation with respect to position coordinates and therefore commutes with the integration over T . Since k depends solely on T (which may in turn depend on the position coordinates), we must differentiate with respect to T and then perform the gradient operation on T , as shown above.

Next, since ∇T depends on position coordinates, and not on T , the ∇T may be removed from the integral sign

$$\nabla F = \nabla T \int \frac{dk}{dT} dT$$

When the integration is performed, we get

$$\nabla F = k \nabla T$$

We now form the Laplacian of F

$$\nabla^2 F = (\nabla \cdot \nabla F) = (\nabla \cdot k \nabla T)$$

But $(\nabla \cdot k \nabla T) = 0$ by the equation of energy. Therefore, we have finally

$$\nabla^2 F = 0$$

11C.5 Effective thermal conductivity of a solid with spherical inclusions

a. At very large distances from the region containing the inclusions, the coordinates of the various inclusions (r, θ) will not be very different from one another. Furthermore, if the density of the inclusions is small, the effect of the various inclusions will be additive. Therefore to get the temperature field far from the region containing the inclusions, we can write

$$T_0(r, \theta) - T^\circ = \left[1 - n \frac{k_1 - k_0}{k_1 + 2k_0} \left(\frac{R}{r} \right)^3 \right] Ar \cos \theta$$

in which n is the number of inclusions, each with a thermal conductivity k_1 . This is the equation that describes the system in Fig. 11C.5(a).

b. For the system in Fig. 11C.5(b), we can apply Eq. 11B.8-2 directly

$$T_0(r, \theta) - T^\circ = \left[1 - \frac{k_{\text{eff}} - k_0}{k_{\text{eff}} + 2k_0} \left(\frac{R'}{r} \right)^3 \right] Ar \cos \theta$$

regarding the shaded sphere as a hypothetical material of thermal conductivity k_{eff} .

c. Next we relate the volume of the inclusions in the true system to the effective volume of the inclusions in the equivalent system: $\frac{4}{3} \pi R^3 n = \frac{4}{3} \pi R'^3 \phi$. Hence $R'^3 = (n/\phi)R^3$.

d. We can now equate the right sides of the above two equations for the temperature profiles far from the origin

$$n \frac{k_1 - k_0}{k_1 + 2k_0} \left(\frac{R}{r} \right)^3 = \frac{k_{\text{eff}} - k_0}{k_{\text{eff}} + 2k_0} \left(\frac{R}{r} \right)^3 \frac{n}{\phi}$$

We can now solve for the effective thermal conductivity, and express the ratio k_{eff}/k_0 as $1 +$ (deviation resulting from the inclusions). This is exactly Eq. 9.6-1.

11C.7 Effect of surface-tension gradients on a falling film

a. If we let I be the gas phase and II be the liquid phase, then the contribution from Eq. 11C.6-4 to the stress component τ_{xz} will be the z-component of

$$[\delta_x \cdot \tau^{\text{II}}] = \nabla^s \sigma \quad \text{or} \quad \tau_{xz}^{\text{II}} = \frac{\partial \sigma}{\partial z}$$

Hence, in Eq. 2.2-13 has to be replaced by

$$\tau_{xz} = (\rho g \cos \beta)x + A$$

When this is combined with Newton's law of viscosity, Eq. 2.2-14, we get

$$-\mu \frac{dv_z}{dx} = (\rho g \cos \beta)x + A$$

When this is integrated, and the no-slip boundary condition at the wall is used, we obtain the velocity profile as follows:

$$v_z = \frac{\rho g \delta^2 \cos \beta}{2\mu} \left[1 - \left(\frac{x}{\delta} \right)^2 \right] + \frac{A\delta}{\mu} \left(1 - \frac{x}{\delta} \right)$$

b. The mass rate of flow in the film is

$$\begin{aligned} w &= \int_0^W \int_0^\delta \rho v_z dx dy = \rho W \delta \int_0^1 v_z d\xi \\ &= \rho W \delta \left[\int_0^1 \frac{\rho g \delta^2 \cos \beta}{2\mu} (1 - \xi^2) d\xi + \frac{A\delta}{\mu} \int_0^1 (1 - \xi) d\xi \right] \\ &= \frac{\rho^2 W g \delta^3}{3\mu} + \frac{A \rho W \delta^2}{2\mu} \end{aligned}$$

11D.1 Equation of change for entropy

a. For the volume element $\Delta x \Delta y \Delta z$ we write a balance equation (not a conservation equation) for the entropy:

$$\begin{aligned} \frac{\partial}{\partial t} (\rho \hat{S} \Delta x \Delta y \Delta z) &= (\rho \hat{S} v_x) \Big|_x \Delta y \Delta z - (\rho \hat{S} v_x) \Big|_{x+\Delta x} \Delta y \Delta z + \dots \\ &\quad + s_x \Big|_x \Delta y \Delta z - s_x \Big|_{x+\Delta x} \Delta y \Delta z + \dots + g_s \Delta x \Delta y \Delta z \end{aligned}$$

Here the dots indicate the additional terms associated with the transport in the y and z directions. Then after dividing by $\Delta x \Delta y \Delta z$ and taking the limit as the volume element goes to zero, we get:

$$\frac{\partial}{\partial t} \rho \hat{S} = - \left(\frac{\partial}{\partial x} \rho \hat{S} v_x + \dots \right) - \left(\frac{\partial}{\partial x} s_x + \dots \right) + g_s$$

which is equivalent to Eq. 11D.1-1.

b. Along a streamline we can write (using Eq. 11.2-2)

$$\frac{D\hat{U}}{Dt} = T \frac{D\hat{S}}{Dt} - p \frac{D\hat{V}}{Dt} \quad \text{or} \quad \frac{1}{\rho} [-(\nabla \cdot \mathbf{q}) - p(\nabla \cdot \mathbf{v}) - (\boldsymbol{\tau} : \nabla \mathbf{v})] = T \frac{D\hat{S}}{Dt} - p \frac{D}{Dt} \frac{1}{\rho}$$

The use of the equation of continuity and subsequent multiplication by ρ/T leads to Eq. 11D.1-3.

c. Insertion of $\mathbf{s} = \mathbf{q}/T$ into Eq. 11D.1-3 leads to

$$\begin{aligned} \rho \frac{D\hat{S}}{Dt} &= -\frac{1}{T} (\nabla \cdot T\mathbf{s}) - \frac{1}{T} (\boldsymbol{\tau} : \nabla \mathbf{v}) = -(\nabla \cdot \mathbf{s}) - \left(\frac{\mathbf{s}}{T} \cdot \nabla T \right) - \frac{1}{T} (\boldsymbol{\tau} : \nabla \mathbf{v}) \\ &= -(\nabla \cdot \mathbf{s}) - \frac{1}{T^2} (\mathbf{q} \cdot \nabla T) - \frac{1}{T} (\boldsymbol{\tau} : \nabla \mathbf{v}) \end{aligned}$$

In the last form, the right side of the equation has the same form as Eq. 11D.1-2, so that the last two terms can be identified as the rate of entropy production (as displayed in Eq. 11D.1-4).

11D.2 Viscous heating in laminar tube flow

a. We introduce the following reduced variables:

$$\xi = \frac{r}{R} \quad \zeta = \frac{\alpha z}{R^2 v_{\max}} \quad \Theta = \frac{k(T - T^{\circ})}{\mu v_{\max}^2}$$

Here T° is the wall temperature (and entry temperature) for the isothermal wall problem, and the entry temperature for the adiabatic (insulated) wall problem. Then Eq. 11B.2-1 may be written in dimensionless form as

$$(1 - \xi^2) \frac{\partial \Theta}{\partial \zeta} = \frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial \Theta}{\partial \xi} \right) + 4\xi^2 \quad (*)$$

The boundary conditions are:

$$\begin{array}{ll} \text{Isothermal wall:} & \text{at } \zeta = 0, \quad \Theta = 0 \\ & \text{at } \xi = 0, \quad \partial \Theta / \partial \xi = 0 \\ & \text{at } \xi = 1, \quad \Theta = 0 \end{array}$$

$$\begin{array}{ll} \text{Insulated wall:} & \text{at } \zeta = 0, \quad \Theta = 0 \\ & \text{at } \xi = 0, \quad \partial \Theta / \partial \xi = 0 \\ & \text{at } \xi = 1, \quad \partial \Theta / \partial \xi = 0 \end{array}$$

The solution to the differential equation will be written, for both cases, as a sum of two functions: $\Theta = \Theta_1 + \Theta_2$. Here the function Θ_1 is that derived in Problem 11B.2:

$$\text{Isothermal wall:} \quad \Theta_1 = \frac{1}{4}(1 - \xi^4)$$

$$\text{Insulated wall:} \quad \Theta_1 = 4\zeta + \left(\xi^2 - \frac{1}{2}\xi^4 \right)$$

When these expressions are put into the partial differential equation (*), we then get for *both* cases

$$\text{Isothermal and insulated wall:} \quad (1 - \xi^2) \frac{\partial \Theta_2}{\partial \zeta} = \frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial \Theta_2}{\partial \xi} \right)$$

This problem is solved by the method of separation of variables, and the solution is of the form:

$$\Theta_2(\xi, \zeta) = -\sum_{i=1}^{\infty} B_i \phi_i(\xi) \psi_i(\zeta)$$

in which the B_i are constants, and the functions $\phi_i(\xi)$ and $\psi_i(\zeta)$ must satisfy the following differential equations:

$$\frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{d\phi_i}{d\xi} \right) + a_i (1 - \xi^2) \phi_i \quad (**) \quad \frac{d\psi_i}{d\zeta} = -a_i \psi_i$$

The a_i are the eigenvalues of the problem, obtained when the solutions to the ϕ_i equation are required to satisfy the boundary conditions. The solution to the ψ_i are proportional to $\exp(-a_i \zeta)$. The solution to the ϕ_i equation has to be done by a power-series expansion, and we return to that shortly. For the moment we remark that the solutions to the ϕ_i equation must satisfy an orthogonality relation:

$$\int_0^1 \phi_i \phi_j (1 - \xi^2) \xi d\xi = 0 \quad \text{for } i \neq j$$

The final temperature distribution expressions are then

$$\text{Isothermal wall:} \quad \Theta = \frac{1}{4} (1 - \xi^4) - \sum_{i=1}^{\infty} B_i \phi_i(\xi) \exp(-a_i \zeta)$$

$$\text{Insulated wall:} \quad \Theta = 4\zeta + \left(\xi^2 - \frac{1}{2} \xi^4 \right) - \sum_{i=1}^{\infty} B_i \phi_i(\xi) \exp(-a_i \zeta)$$

From these solutions, it can be seen that the solutions obtained in Problem 11B.2 are just the limiting solutions for large values of the dimensionless axial variable ζ .

The constants B_i are obtained by writing the above equations for $\zeta = 0$, then multiplying by $\phi_j(1 - \xi^2)\xi$, and next integrating over ξ from 0 to 1. Doing that we get:

$$\text{Isothermal wall: } B_i = \frac{1}{4} \frac{\int_0^1 (1 - \xi^4) \phi_j(1 - \xi^2) \xi d\xi}{\int_0^1 \phi_i^2(1 - \xi^2) \xi d\xi}$$

$$\text{Insulated wall: } B_i = \frac{\int_0^1 (\xi^2 - \frac{1}{2} \xi^4) \phi_j(1 - \xi^2) \xi d\xi}{\int_0^1 \phi_i^2(1 - \xi^2) \xi d\xi}$$

The above results were first given by H. C. Brinkman, *Appl. Sci. Res.*, **A2**, 120-124 (1951).

The eigenfunctions $\phi_i(\xi)$ that satisfy (**) can be written as a series expansion:

$$\phi_i(\xi) = \sum_{k=0}^{\infty} b_{ik} \xi^k$$

The coefficient b_{i0} may be arbitrarily chosen to be unity. The next coefficient, b_{i1} , must be chosen to be zero in order to satisfy the boundary condition at the tube axis. Substitution of the above series solution into (**) gives the following recursion formula:

$$b_{ik} = -\frac{a_i}{k^2} (b_{i,k-2} - b_{i,k-5})$$

Therefore, all of the b_{ik} can be expressed in terms of the eigenvalues a_i as follows:

$$\begin{array}{llll} b_{i0} = 1 & b_{i2} = -\frac{1}{4} a_i & b_{i4} = +\frac{1}{64} a_i^2 & b_{i6} = +\frac{1}{2304} a_i^3 & \text{etc.} \\ b_{i1} = 0 & b_{i3} = 0 & b_{i5} = +\frac{1}{25} a_i & b_{i7} = -\frac{29}{4900} a_i^2 & \end{array}$$

The eigenvalues a_i are determined from the boundary conditions at $\xi = 1$, which require that:

Isothermal wall:
$$\sum_{k=0}^{\infty} b_{ik} = 0$$

Insulated wall:
$$\sum_{k=0}^{\infty} kb_{ik} = 0$$

The left side of each of these equations is a polynomial in a_i . Setting the polynomial expression equal to zero gives an algebraic equation which has to be solved for the infinite set of a_i . This is clearly a very tedious process. The first few eigenvalues have been calculated by Brinkman in the reference cited above. Also given there are some sample temperature profiles.

b. The power-law fluid can be handled by the same method used for the Newtonian fluid, given in part (a). The reference to Bird, cited on p. 373, gives the details and numerical results.

11D.3 Derivation of the energy equation using integral theorems

a. For an arbitrary volume V fixed in space, we can write the total energy balance as

$$\frac{d}{dt} \int_V \left(\rho \hat{U} + \frac{1}{2} \rho v^2 \right) dV = - \int_S (\mathbf{n} \cdot \mathbf{e}) dS + \int_V (\mathbf{v} \cdot \mathbf{g}) dV$$

$$\int_V \frac{\partial}{\partial t} \left(\rho \hat{U} + \frac{1}{2} \rho v^2 \right) dV = - \int_V (\nabla \cdot \mathbf{e}) dV + \int_V (\mathbf{v} \cdot \mathbf{g}) dV$$

To get the second form, the Gauss divergence theorem was used on the right side, and on the left side the time derivative is taken inside the integral (because the volume element is fixed). Then, since the volume V was chosen arbitrarily, the integral signs may be removed to get Eq. 11.1-6.

b. For an arbitrary moving volume element $V(t)$ of fluid, we may write the equation of conservation of energy as follows:

$$\frac{d}{dt} \int_{V(t)} \left(\rho \hat{U} + \frac{1}{2} \rho v^2 \right) dV = - \int_{S(t)} (\mathbf{n} \cdot \mathbf{q}) dS - \int_{S(t)} (\mathbf{n} \cdot [\boldsymbol{\pi} \cdot \mathbf{v}]) dS + \int_{V(t)} (\mathbf{v} \cdot \mathbf{g}) dV$$

This accounts for the heat transported into V across the surface, and the work done on V by the surface forces. There is no term accounting for convective transport across S , since the surface itself is moving with the fluid velocity. Next, use the Leibnitz formula on the left side to get

$$\frac{d}{dt} \int_{V(t)} \left(\rho \hat{U} + \frac{1}{2} \rho v^2 \right) dV = \int_{V(t)} \frac{\partial}{\partial t} \left(\rho \hat{U} + \frac{1}{2} \rho v^2 \right) dV + \int_{S(t)} \left(\rho \hat{U} + \frac{1}{2} \rho v^2 \right) (\mathbf{n} \cdot \mathbf{v}_S) dS$$

$$\int_{V(t)} \frac{\partial}{\partial t} \left(\rho \hat{U} + \frac{1}{2} \rho v^2 \right) dV + \int_{S(t)} \left(\mathbf{n} \cdot \left(\rho \hat{U} + \frac{1}{2} \rho v^2 \right) \mathbf{v}_S \right) dS$$

The surface velocity \mathbf{v}_S for a moving blob of fluid is identical to the fluid velocity \mathbf{v} . When the term containing $\mathbf{v}_S = \mathbf{v}$ is transferred to the right side and the Gauss divergence theorem is used, we get the energy balance in the following form

$$\int_{V(t)} \frac{\partial}{\partial t} (\rho \hat{U} + \frac{1}{2} \rho v^2) dV = - \int_{V(t)} (\nabla \cdot (\rho \hat{U} + \frac{1}{2} \rho v^2) \mathbf{v}) dV$$

$$- \int_{V(t)} (\nabla \cdot \mathbf{q}) dV - \int_{V(t)} (\nabla \cdot [\boldsymbol{\pi} \cdot \mathbf{v}]) dV + \int_{V(t)} (\mathbf{v} \cdot \mathbf{g}) dV$$

Since the element of volume was completely arbitrary, we may now remove the integral signs through the entire equation and obtain Eq. 11.1-7.

12A.1 Unsteady-state heat conduction in an iron sphere

a. The thermal diffusivity of the sphere is given by Eq. 9.1-8:

$$\alpha = \frac{k}{\rho \hat{C}_p} = \frac{30}{(436)(0.12)} = 0.573 \text{ ft}^2 / \text{hr}$$

b. The center temperature is to be 128°F; hence

$$\frac{T_{\text{ctr}} - T_0}{T_1 - T_0} = \frac{128 - 70}{270 - 70} = 0.29$$

Then, from Fig. 12.1-3, $\alpha t / R^2 = 0.1$, and

$$t = 0.1 \left(\frac{R^2}{\alpha} \right) = 0.1 \left(\frac{(1/24)^2}{0.573} \right) = 3.03 \times 10^{-4} \text{ hrs} = 1.1 \text{ s}$$

c. By equating the dimensionless times, we get

$$\frac{\alpha_1 t_1}{R_1^2} = \frac{\alpha_2 t_2}{R_2^2}$$

or

$$\alpha_2 = \alpha_1 \left(\frac{t_1}{t_2} \right) = 0.573 \left(\frac{1}{2} \right) = 0.287$$

d. The partial differential equation from which Fig. 12.1-3 was constructed is

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right)$$

12A.2 Comparison of the two slab solutions for short times

According to Figure 12.1-1, at $\alpha t/b^2 = 0.01$ and $y/b = 0.9$

$$\frac{T - T_0}{T_1 - T_0} \approx 0.46$$

where y is the distance from the mid-plane of the slab.

Next we use Fig. 4.1-1, which can be interpreted as a plot of $(T - T_0)/(T_1 - T_0)$ vs $y'/\sqrt{4\alpha t}$, where $y' = b - y$ is the distance from the wall. We then get

$$\frac{y'}{\sqrt{4\alpha t}} = \frac{1(1-0.9)}{2\sqrt{\alpha t/b^2}} = \frac{1}{2}$$

Then from Fig. 4.1-1 we find

$$\frac{T - T_0}{T_1 - T_0} \approx 0.48$$

Hence the use of the combination of variables solution introduces an error of about 4%. Smaller errors occur at smaller values of the dimensionless time $\alpha t/b^2$.

12A.3 Bonding with a thermosetting adhesive

The dimensionless temperature at the time of bonding is

$$\frac{T_{\text{ctr}} - T_0}{T_1 - T_0} = \frac{160 - 20}{220 - 20} = 0.70$$

This occurs very nearly at a dimensionless temperature $\alpha t/b^2 = 0.6$.
Hence the time required is

$$t = (0.6) \left(\frac{b^2}{\alpha} \right) = (0.6) \left(\frac{(0.77)^2}{4.2 \times 10^{-3}} \right) = 85\text{s}$$

12A.4 Quenching of a steel billet

The thermal diffusivity of the steel billet is

$$\alpha = \frac{k}{\rho \hat{C}_p} = \frac{(25)(4.1365 \times 10^{-3})}{(7.7)(0.12)} = 0.112 \text{ cm}^2/\text{s}$$

The dimensionless time is then

$$\frac{\alpha t}{R^2} = \frac{(0.112)(5 \times 60)}{(6 \times 2.54)^2} = 0.145$$

From Fig. 12.1-2, the dimensionless center-line temperature is about 0.31. Therefore

$$\frac{T_{\text{ctr}} - T_0}{T_1 - T_0} = 0.31$$

We can now solve for the centerline temperature

$$T_{\text{ctr}} = 0.31(T_1 - T_0) + T_0 = 0.31(200 - 1000) + 1000 \approx 750^\circ\text{F}$$

12A.5 Measurement of thermal diffusivity from amplitude of temperature oscillations

a. From Eq. 12.1-40, we see that the amplitude of the

$$A(y) = \frac{q_0}{k} \sqrt{\frac{\alpha}{\omega}} \exp\left(-\sqrt{\frac{\omega}{2\alpha}} y\right)$$

Then the ratio of amplitudes at two different distances from the plane $y = 0$ is

$$\frac{A_1}{A_2} = \exp\left(-\sqrt{\frac{\omega}{2\alpha}} (y_1 - y_2)\right)$$

The logarithm of this expression is

$$\ln\left(\frac{A_1}{A_2}\right) = -\sqrt{\frac{\omega}{2\alpha}} (y_1 - y_2) = +\sqrt{\frac{\omega}{2\alpha}} (y_2 - y_1)$$

This can be solved for the thermal diffusivity to obtain

$$\alpha = \frac{\omega}{2} \left(\frac{y_2 - y_1}{\ln(A_1/A_2)} \right)^2 = \frac{2\pi\nu}{2} \left(\frac{y_2 - y_1}{\ln(A_1/A_2)} \right)^2 = \pi\nu \left(\frac{y_2 - y_1}{\ln(A_1/A_2)} \right)^2$$

12A.6 Forced convection from a sphere in creeping flow

a. The dimensions of the quantities in Eq. 12.4-34 can be obtained from the table on pp. 872-876, thus:

$$Q[=] \frac{ML^2}{t^3} \quad D[=]L \quad T[=]T \quad k[=] \frac{ML}{t^3T}$$

$$\text{Re}[=] \text{dimensionless} \quad \text{Pr}[=] \text{dimensionless}$$

To check for dimensional consistency:

$$\frac{ML^2}{t^3} [=] L \cdot T \cdot \frac{ML}{t^3T}$$

b. We now write Eq. 12.4-34 in terms of the Péclet number:

$$\begin{aligned} Q &= (\pi D^2)(T_0 - T_\infty) \left(\frac{k}{D}\right) \left[\frac{(3\pi)^{2/3}}{2^{7/3} \Gamma(\frac{4}{3})} \right] \left(\frac{Dv_\infty \rho \hat{C}_p}{k} \right)^{1/3} \\ &= (\pi D)(T_0 - T_\infty) k \left[\frac{(3\pi)^{2/3}}{2^{7/3} \Gamma(\frac{4}{3})} \right] \left(\frac{Dv_\infty \rho \hat{C}_p}{k} \right)^{1/3} \end{aligned}$$

In this form, the viscosity does not appear in the expression. We now fill in the numerical values, using c.g.s. units throughout:

$$\begin{aligned} Q &= \pi(0.1)(50)(3 \times 10^{-4}) \left[\frac{4.462}{(5.04)(0.89297)} \right] \left(\frac{(0.1)(1.0)(0.9)(0.45)}{3 \times 10^{-4}} \right)^{1/3} \\ &= 0.0240 \text{ cal/s} \end{aligned}$$

12B.1 Measurement of thermal diffusivity in an unsteady-state experiment

We first tabulate the midplane temperatures as a function of time, and then convert them to the dimensionless midplane temperatures. Having done this, we consult Figure 12.1-1. We find

of the graph, and from this we can read off the value of $\alpha t/b^2$. This quantity then can be multiplied by b^2 and divided by t to get the thermal diffusivity. The calculations may be summarized as follows (working to two significant figures):

t	T_{midplane}	$\frac{T_{\text{midplane}} - T_0}{T_1 - T_0}$	$\alpha t/b^2$	α
0	20.0	0		
120	24.4	0.220	0.20	1.50×10^{-3}
240	28.5	0.525	0.40	1.50×10^{-3}

12B.2 Two-dimensional forced convection with a line heat source

a. The energy equation simplifies to

$$\rho \hat{C}_p v_x \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial y^2} \quad \text{where } v_x = v_0, \text{ a constant}$$

The boundary conditions have the following meanings:

Eq. 12B.2-1: Far from the wire, the temperature of the fluid is unchanged from its value, T_∞ , for $x < 0$

Eq. 12B.2-2: The approaching fluid is all at the temperature T_∞ --that is, there is no heat conduction upstream

Eq. 12B.2-3: The entering into the fluid from the wire must appear somewhere in the cross-section

In addition, we need a statement that $T(x, y) = T(x, -y)$ --that is, symmetry about the plane $y = 0$, which contains the wire.

b. The postulated solution in Eq. 12B.2-4 states that the temperature profile at any value of x will be geometrically similar to the profile at any other value of x . When this expression for T is inserted into Eq. 12B.2-3 we get:

$$\rho \hat{C}_p v_0 \int_{-\infty}^{+\infty} f(x) g(\eta) \delta(x) d\eta = Q/L$$

When the integral on η is evaluated, it is found that $f(x)\delta(x) = C_1$, a constant.

When the expression in Eq. 12B.2-4 is inserted into the energy equation, we get

$$\frac{v_0}{\alpha} \frac{\partial}{\partial x} [f(x)g(\eta)] = \frac{\partial^2}{\partial y^2} [f(x)g(\eta)], \text{ or}$$

$$\frac{v_0}{\alpha} \frac{\partial}{\partial x} \left[\frac{1}{\delta(x)} g\left(\frac{y}{\delta(x)}\right) \right] = \frac{\partial^2}{\partial y^2} \left[\frac{1}{\delta(x)} g\left(\frac{y}{\delta(x)}\right) \right], \text{ or}$$

$$\frac{v_0}{\alpha} \left[-\frac{1}{\delta^2} \frac{d\delta}{dx} g - \frac{\eta}{\delta^2} \frac{dg}{d\eta} \frac{d\delta}{dx} \right] = \frac{1}{\delta} \frac{d^2 g}{d\eta^2} \frac{1}{\delta^2}$$

Multiplication by δ^3 then gives Eq. 12B.2-5.

c. When the quantity in brackets in Eq. 12B.2-5 is set equal to 2, we get a first-order separable equation for $\delta(x)$ --the thermal boundary-layer thickness--which is integrated from 0 to x to give $\delta(x) = \sqrt{4\alpha x/v_0}$. The main reason for setting the bracketed quantity equal to 2 is that the solution to the equation for $g(\eta)$ comes out to be a little simpler.

d. The equation for $g(\eta)$ can be written as

$$\frac{d}{d\eta} \left(\frac{dg}{d\eta} \right) = -2 \frac{d}{d\eta} (\eta g)$$

which has the solution

$$\frac{dg}{d\eta} = -2\eta g + C_2$$

Since $g(\eta)$ is symmetric about $\eta = 0$, we know that $dg/d\eta = 0$ at $\eta = 0$; therefore, this last equation tells us that $C_2 = 0$. A further integration leads to a Gaussian function

$$g = C_3 e^{-\eta^2}$$

We do not evaluate C_3 but instead "absorb" into the constant C_1 .

e. We can now evaluate C_1 by substituting the postulated temperature profile into Eq. 12B.2-3 to get

$$\rho \hat{C}_p v_0 C_1 \int_{-\infty}^{+\infty} g(\eta) d\eta = Q/L \quad \text{or} \quad \rho \hat{C}_p v_0 C_1 \int_{-\infty}^{+\infty} e^{-\eta^2} d\eta = Q/L$$

which gives

$$C_1 = \frac{Q/L}{\rho \hat{C}_p v_0 \sqrt{\pi}}$$

Which then completes the determination of the temperature distribution in the wake of the wire.

12B.3 Heating of a wall (constant heat flux)

a. The equation to be solved for $q_y(y, t)$ is

$$\frac{\partial q_y}{\partial t} = \alpha \frac{\partial^2 q_y}{\partial y^2} \quad \text{with } q_y(y, 0) = 0, \quad q_y(0, t) = q_0, \quad \text{and } q_y(\infty, t) = 0$$

This is the same mathematical problem that is solved in Example 12.1-1, so that we can write down at once

$$\frac{q_y}{q_0} = \frac{2}{\sqrt{\pi}} \int_{y/\sqrt{4\alpha t}}^{\infty} e^{-u^2} du$$

To get the temperature we use Eq. 12.1-39

$$\begin{aligned} T - T_0 &= \frac{q_0}{k} \int_y^{\infty} \frac{q_y(\bar{y}, t)}{q_0} d\bar{y} = \frac{q_0}{k} \frac{2}{\sqrt{\pi}} \int_y^{\infty} \int_{\bar{y}/\sqrt{4\alpha t}}^{\infty} e^{-u^2} du d\bar{y} \\ &= \frac{q_0}{k} \frac{2}{\sqrt{\pi}} \sqrt{4\alpha t} \int_{y/\sqrt{4\alpha t}}^{\infty} \int_Y^{\infty} e^{-u^2} du dY = \frac{q_0}{k} \frac{2}{\sqrt{\pi}} \sqrt{4\alpha t} \int_{y/\sqrt{4\alpha t}}^{\infty} \int_{y/\sqrt{4\alpha t}}^u e^{-u^2} dY du \\ &= \frac{q_0}{k} \frac{2}{\sqrt{\pi}} \sqrt{4\alpha t} \int_{y/\sqrt{4\alpha t}}^{\infty} e^{-u^2} \left[u - \frac{y}{\sqrt{4\alpha t}} \right] du \\ &= \frac{q_0}{k} \left(\sqrt{\frac{4\alpha t}{\pi}} \int_{y/\sqrt{4\alpha t}}^{\infty} e^{-u^2} u du - \frac{2y}{\sqrt{\pi}} \int_{y/\sqrt{4\alpha t}}^{\infty} e^{-u^2} du \right) \end{aligned}$$

When the first integral is evaluated, we get Eq. 12B.3-1. In the above, to get the second line, we made a change of variables and then interchanged the order of integration; in the third line, we performed the inner integration.

b. Some intermediate steps in showing that the partial differential equation is satisfied: We write $k(T - T_0)/q_0 = F - G$. Then

$$\frac{\partial F}{\partial t} = \left(\sqrt{\frac{\alpha}{\pi t}} + \sqrt{\frac{1}{4\pi\alpha t}} \frac{y^2}{t} \right) e^{-y^2/4\alpha t}; \quad \alpha \frac{\partial^2 F}{\partial y^2} = \left(\sqrt{\frac{1}{4\pi\alpha t}} \frac{y^2}{t} - \sqrt{\frac{\alpha}{\pi t}} \right) e^{-y^2/4\alpha t}$$

$$\frac{\partial G}{\partial t} = \left(\sqrt{\frac{1}{4\pi\alpha t}} \frac{y^2}{t} \right) e^{-y^2/4\alpha t}; \quad \alpha \frac{\partial^2 G}{\partial y^2} = \left(-2\sqrt{\frac{\alpha}{\pi t}} + \sqrt{\frac{1}{4\pi\alpha t}} \frac{y^2}{t} \right) e^{-y^2/4\alpha t}$$

12B.4 Heat transfer from a wall to a falling film (short contact time limit)

a. From Eq. 2.2-18, we get

$$\begin{aligned} v_z &= v_{z,\max} \left[1 - (x/\delta)^2 \right] = v_{z,\max} \left[1 - (1 - (y/\delta))^2 \right] \\ &= v_{z,\max} \left[1 - 1 + 2(y/\delta) + (y/\delta)^2 \right] \rightarrow 2v_{z,\max} (y/\delta) \end{aligned}$$

this last expression is good in the vicinity of the wall, where the quadratic term can be neglected.

b. Equation 12B.4-2 presupposes that the heat conduction in the z direction can be neglected relative to the heat convection in the z direction. In addition, laminar, nonrippling flow is assumed.

c. The fictitious boundary condition at an infinite distance from the wall may be used instead of the boundary condition at a distance δ from the wall, since for short contact times the fluid is heated over a very short distance y . Therefore the infinite boundary condition can be expected to be adequate.

d. Equation 12B.4-3 can be written as $y(\partial\Theta/\partial z) = \beta(\partial^2\Theta/\partial y^2)$. Next we have to convert the derivatives to derivatives with respect to the dimensionless variable η :

$$\frac{\partial\Theta}{\partial z} = \frac{d\Theta}{d\eta} \frac{\partial\eta}{\partial z} = \frac{d\Theta}{d\eta} \frac{y}{\sqrt[3]{9\beta z}} \left(-\frac{1}{3z} \right)$$

$$\frac{\partial\Theta}{\partial y} = \frac{d\Theta}{d\eta} \frac{\partial\eta}{\partial y} = \frac{d\Theta}{d\eta} \frac{1}{\sqrt[3]{9\beta z}}; \quad \frac{\partial^2\Theta}{\partial y^2} = \frac{d}{d\eta} \left(\frac{d\Theta}{d\eta} \frac{1}{\sqrt[3]{9\beta z}} \right) \frac{\partial\eta}{\partial y} = \frac{d^2\Theta}{d\eta^2} \left(\frac{1}{\sqrt[3]{9\beta z}} \right)^2$$

When these relations are substituted into the partial differential equation and use is made of the defining equation for η we get Eq. 12B.4-7.

e. When we set $d\Theta/d\eta = p$, we get $dp/d\eta + 3\eta^2 p = 0$, which is first-order and separable, and the solution is given in the book. The next integration gives

$$\Theta = C_1 \int_0^\eta e^{-\bar{\eta}^3} d\bar{\eta} + C_2$$

12B.5 Temperature in a slab with heat production

a. This problem is discussed on pp. 130-131 of the 2nd Edition of Carslaw and Jaeger. The solution in dimensionless form can be obtained from Eq. (7) on p. 130. We must first determine the correspondence between their symbols and ours:

C&J	l	K	v	x/l	κ	$\kappa t/l^2$	A_0
BS&L	b	k	$T - T_0$	$\eta = \frac{y}{b}$	$\alpha = \frac{k}{\rho \hat{C}_p}$	$\tau = \frac{\alpha t}{b^2}$	S_0

Therefore the temperature rise as a function of position and time is

$$\frac{k(T - T_0)}{S_0 b^2} = \frac{1}{2} \left(1 - \eta^2 - 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + \frac{1}{2})^3 \pi^3} \cos(n + \frac{1}{2})\pi\eta \cdot e^{-(n + \frac{1}{2})^2 \pi^2 \tau} \right)$$

b, The center-plane temperature is obtained by setting η equal to zero. The maximum of the center-plane temperature is then

$$T_{\max} = T_0 + \frac{S_0 b^2}{2k}$$

c. According to the figure on p. 131 in Carslaw and Jaeger, 90% of the total temperature rise occurs at about $\tau = 1$, that is, at about $t = b^2/\alpha$.

12B.6 Forced convection in slow flow across a cylinder

a. First we have to determine the velocity component v_θ from Eq. 12B.6-1 (here we let $\xi = r/R$ and measure θ downstream from the stagnation locus):

$$v_\theta = \frac{\partial \psi}{\partial r} = \frac{v_\infty \sin \theta}{2S} \left[(2 \ln \xi - 1) + 2 - \frac{1}{\xi^2} \right]$$

Then let $\xi = 1 + \eta$ and, for small η , introduce the Taylor expansions $\ln \xi = (\xi - 1) + \dots = \eta + \dots = y/R + \dots$ and $-1/\xi^2 = -1 + 2\eta + \dots$. Then we may write

$$v_x = \frac{v_\infty \sin \theta}{2S} [4\eta + \dots] \approx \frac{2v_\infty \sin \theta}{SR} y \equiv \beta y$$

Therefore, $\beta = 2v_\infty \sin \theta / SR$.

b. We identify the boundary-layer coordinates as $x = \theta$, $y = r - R$, and $z = z$. Then we recognize that $h_x = R$ and set $h_z = 1$. Therefore we can get the heat loss from a length L of the cylinder as follows, starting from Eq. 12.4-31:

$$\begin{aligned} Q &= \frac{3^{1/3} k (T_0 - T_\infty) L \left(\frac{2v_\infty}{SR} \right)^{1/3} \left(\int_0^\pi \sqrt{\sin \theta} R d\theta \right)^{2/3}}{2\alpha^{1/3} \Gamma\left(\frac{4}{3}\right)} \\ &= \frac{3^{1/3} k (T_0 - T_\infty) L \left(\frac{2v_\infty}{SR} \right)^{1/3} \left(B\left(\frac{3}{4}, \frac{1}{2}\right) \right)^{2/3} R^{2/3}}{2\alpha^{1/3} \Gamma\left(\frac{4}{3}\right)} \\ &= \frac{3^{1/3} \left(B\left(\frac{3}{4}, \frac{1}{2}\right) \right)^{2/3}}{2\pi \Gamma\left(\frac{4}{3}\right) S^{1/3}} (\pi DL) (T_0 - T_\infty) \left(\frac{k}{D} \right) \left(\frac{Dv_\infty}{\nu} \cdot \frac{\nu}{\alpha} \right)^{1/3} \\ &= \frac{3^{1/3} \left(B\left(\frac{3}{4}, \frac{1}{2}\right) \right)^{2/3}}{2\pi \Gamma\left(\frac{4}{3}\right)} (\pi DL) (T_0 - T_\infty) \left(\frac{k}{D} \right) \left(\frac{\text{Re Pr}}{S} \right)^{1/3} \end{aligned}$$

Comparison of this result with the solution to part (b) given in the text allows the constant C to be evaluated.

c. The boundary-layer thickness can be obtained from Eq. 12.4-29 as follows:

$$\begin{aligned} \frac{\delta_T}{R} &= \frac{1}{R} \sqrt{\frac{RS}{2v_\infty \sin \theta}} \left(9\alpha \int_0^\theta \sqrt{\frac{2v_\infty \sin \theta}{RS}} R d\theta \right)^{1/3} \\ &= \left(\frac{9S}{\text{Re Pr}} \right)^{1/3} \frac{1}{\sqrt{\sin \theta}} \left(\int_0^\theta \sqrt{\sin \theta} d\theta \right)^{1/3} \\ &= \left(\frac{9S}{\text{Re Pr}} \right)^{1/3} f(\theta) \end{aligned}$$

At the separation locus, $\theta = \pi$, the integral is finite (according to Eq. 12B.6-2) and the denominator in f is zero, so that f is infinite. Near the stagnation locus, $\theta = 0$, the numerator and denominator of f may be expanded in Taylor series to give:

$$\begin{aligned} f &= \frac{\left(\int_0^\theta \sqrt{\theta - \frac{1}{6}\theta^3 + \dots} d\theta \right)^{1/3}}{\sqrt{\theta - \frac{1}{6}\theta^3 + \dots}} = \frac{\left(\int_0^\theta \sqrt{\theta} \left(1 - \frac{1}{12}\theta^2 + \dots \right) d\theta \right)^{1/3}}{\sqrt{\theta} \left(1 - \frac{1}{12}\theta^2 + \dots \right)} \\ &= \frac{1}{\sqrt{\theta}} \left(1 + \frac{1}{12}\theta^2 + \dots \right) \left(\frac{2}{3}\theta^{3/2} - \frac{1}{12} \frac{2}{7}\theta^{7/2} + \dots \right)^{1/3} \end{aligned}$$

Then in the limit as $\theta \rightarrow 0$, this gives the stagnation value

$$f = \left(\frac{2}{3} \right)^{1/3}$$

Numerical integration gives for $\theta = \frac{1}{2}\pi$

$$f = 1.1981$$

To get the answers given in the book, we have to recognize that the theta in the problem is being measured from the stagnation point at $\theta = \pi$ to the separation point at $\theta = 0$.

Note that for this flow, just as for the flow around the sphere, the boundary layer thickness increases through finite values to an infinite value at the separation locus.

[Note: Eq. 6.2.1 of Abramowitz and Stegun (NBS, Applied Math Series, 55) is a convenient formula from which the integral from zero to $\frac{1}{2}\pi$ can be calculated in terms of gamma functions, which are tabulated.]

12B.9 Non-Newtonian heat transfer with constant wall heat flux (asymptotic solution for small axial distances)

For Newtonian fluids, Example 12.2-2, we wrote

$$\begin{aligned} v_z &= v_{z,\max} \left[1 - \left(\frac{r}{R} \right)^2 \right] = v_{z,\max} \left[1 - \left(1 - \left(\frac{y}{R} \right)^2 \right) \right] = \\ &= v_{z,\max} \left(2 \frac{y}{R} - \left(\frac{y}{R} \right)^2 \right) \approx 2v_{z,\max} \frac{y}{R} = \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{2\mu L} \right) \frac{y}{R} \equiv v_0 \frac{y}{R} \end{aligned}$$

where Eqs. 2.3-18 and 19 have been used, as well as the definition $y = R - r$.

The analogous procedure for power-law fluids gives

$$\begin{aligned} v_z &= v_{z,\max} \left[1 - \left(\frac{r}{R} \right)^{(1/n)+1} \right] = v_{z,\max} \left[1 - \left(1 - \frac{y}{R} \right)^{(1/n)+1} \right] \\ &= v_{z,\max} \left[1 - 1 + \left(\frac{1}{n} + 1 \right) \left(\frac{y}{R} \right) - \frac{1}{2!} \left(\frac{1}{n} + 1 \right) \left(\frac{1}{n} \right) \left(\frac{y}{R} \right)^2 + \dots \right] \\ &\approx v_{z,\max} \left(\frac{1}{n} + 1 \right) \left(\frac{y}{R} \right) = \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{2mL} \right)^{1/n} \frac{R}{(1/n) + 1} \left(\frac{1}{n} + 1 \right) \left(\frac{y}{R} \right) \\ &= \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{2mL} \right)^{1/n} R \left(\frac{y}{R} \right) \equiv v_0 \frac{y}{R} \end{aligned}$$

This gives the formula for the v_0 that has to be used for power-law fluids. The quantity v_0 , with dimensions of velocity, appears only in the dimensionless parameter λ .

The boundary condition at $\eta=0$ gives $C_2=1$. The boundary condition at $\eta=\infty$, gives

$$C_1 = -\frac{1}{\int_0^{\infty} e^{-\bar{\eta}^3} d\bar{\eta}}$$

And the complete solution (in dimensionless form) is

$$\Theta = 1 - \frac{\int_0^{\eta} e^{-\bar{\eta}^3} d\bar{\eta}}{\int_0^{\infty} e^{-\bar{\eta}^3} d\bar{\eta}} = \frac{\int_0^{\infty} e^{-\bar{\eta}^3} d\bar{\eta} - \int_0^{\eta} e^{-\bar{\eta}^3} d\bar{\eta}}{\int_0^{\infty} e^{-\bar{\eta}^3} d\bar{\eta}} = \frac{\int_{\eta}^{\infty} e^{-\bar{\eta}^3} d\bar{\eta}}{\Gamma(\frac{4}{3})}$$

The integral in the numerator cannot be integrated analytically.

f. The local wall heat flux (i.e., at any position z down the wall) is

$$q_y|_{y=0} = -k \frac{\partial T}{\partial y} \Big|_{y=0} = +k \frac{d\Theta}{d\eta} \Big|_{\eta=0} \frac{T_0 - T_1}{\sqrt[3]{9\beta z}} = \frac{k}{\Gamma(\frac{4}{3})} \frac{T_0 - T_1}{\sqrt[3]{9\beta z}}$$

In taking the derivative, the Leibniz formula was used. Finally the average heat flux at the wall is

$$q_{y,avg}|_{y=0} = \frac{1}{L} \int_0^L q_y|_{y=0} dz = \frac{k}{\Gamma(\frac{4}{3})} \frac{T_0 - T_1}{\sqrt[3]{9\beta}} \frac{1}{L} \int_0^L z^{-1/3} dz = \frac{3}{2} \frac{k}{\Gamma(\frac{4}{3})} \frac{T_0 - T_1}{\sqrt[3]{9\beta L}}$$

This is the result that is given in Eq. 12B.4-9.

12C.1 Product solutions for unsteady heat conduction in solids

a. We begin by defining a dimensionless temperature difference by

$$\Delta = \frac{T_1 - T(x, y, z, t)}{T_1 - T_0}$$

in which T_0 is the initial temperature of the solid rectangular parallelepiped, and T_1 is the imposed temperature on the surfaces of the solid. Then the 3-dimensional heat conduction equation for the solid is

$$\frac{\partial \Delta}{\partial t} = \alpha \left(\frac{\partial^2 \Delta}{\partial x^2} + \frac{\partial^2 \Delta}{\partial y^2} + \frac{\partial^2 \Delta}{\partial z^2} \right)$$

If we now postulate a product solution

$$\Delta(x, y, z, t) = X(x, t)Y(y, t)Z(z, t)$$

then we get

$$\frac{\partial (XYZ)}{\partial t} = \alpha \left(\frac{\partial^2 (XYZ)}{\partial x^2} + \frac{\partial^2 (XYZ)}{\partial y^2} + \frac{\partial^2 (XYZ)}{\partial z^2} \right)$$

Division by XYZ then gives

$$\frac{1}{XYZ} \frac{\partial (XYZ)}{\partial t} = \alpha \left(\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} \right)$$

When the product on the left side is differentiated we get

$$\frac{1}{X} \frac{\partial X}{\partial t} + \frac{1}{Y} \frac{\partial Y}{\partial t} + \frac{1}{Z} \frac{\partial Z}{\partial t} = \alpha \left(\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} \right)$$

The first term on the left is a function of x and t , as is the first term on the right. Similarly, the functional dependences of the second and

third terms are the same. Therefore we postulate that these pairs of terms can be equated to give:

$$\frac{\partial X}{\partial t} = \alpha \frac{\partial^2 X}{\partial x^2}; \quad \frac{\partial Y}{\partial t} = \alpha \frac{\partial^2 Y}{\partial y^2}; \quad \frac{\partial Z}{\partial t} = \alpha \frac{\partial^2 Z}{\partial z^2}$$

That is, we get three one-dimensional heat-conduction equations. These can be solved according to the method of separation of variables given in Ex. 12.1-2, and all of them have the same initial conditions, and the same kinds of boundary conditions. Therefore, when we combine the three solutions in the product form above we get

$$\Delta = \left(2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m + \frac{1}{2})\pi} e^{-(m+\frac{1}{2})^2 \pi^2 \alpha t / a^2} \cos(m + \frac{1}{2})\pi x / a \right) \\ \cdot \left(2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + \frac{1}{2})\pi} e^{-(n+\frac{1}{2})^2 \pi^2 \alpha t / b^2} \cos(n + \frac{1}{2})\pi y / b \right) \\ \cdot \left(2 \sum_{p=0}^{\infty} \frac{(-1)^p}{(p + \frac{1}{2})\pi} e^{-(p+\frac{1}{2})^2 \pi^2 \alpha t / c^2} \cos(p + \frac{1}{2})\pi z / c \right)$$

or

$$\Delta = 8 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{m+n+p}}{(m + \frac{1}{2})(n + \frac{1}{2})(p + \frac{1}{2})} \\ \cdot [\cos(m + \frac{1}{2})\pi x / a][\cos(n + \frac{1}{2})\pi y / b][\cos(p + \frac{1}{2})\pi z / c] \\ \cdot e^{-[(m+\frac{1}{2})^2 / a^2 + (n+\frac{1}{2})^2 / b^2 + (p+\frac{1}{2})^2 / c^2] \pi^2 \alpha t}$$

This can be substituted into the differential equation to verify that the latter can indeed be satisfied. Also, the satisfaction of the boundary and initial conditions can be verified.

b. This and many other interesting and important product solutions are discussed in Carslaw and Jaeger on pp. 33, 184, and 227.

12C.2 Heating of a semi-infinite slab with a variable thermal conductivity

a. The heat conduction equation is, for variable k ,

$$\rho \hat{C}_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) \quad \text{or} \quad \rho \hat{C}_p \frac{\partial \Theta}{\partial t} = k_0 \frac{\partial}{\partial y} \left((1 + \beta \Theta) \frac{\partial \Theta}{\partial y} \right) \quad \text{or}$$

$$\frac{\partial \Theta}{\partial t} = \alpha_0 \left[\frac{\partial^2 \Theta}{\partial y^2} + \beta \frac{\partial}{\partial y} \left(\Theta \frac{\partial \Theta}{\partial y} \right) \right]$$

In order to use Eq. 12C.2-2, we need to convert the derivatives:

$$\frac{\partial \Theta}{\partial t} = \frac{d\Phi}{d\eta} \frac{\partial \eta}{\partial t} = \frac{d\Phi}{d\eta} \left(-\frac{y}{\delta^2} \frac{d\delta}{dt} \right) = \frac{d\Phi}{d\eta} \left(-\frac{\eta}{\delta} \frac{d\delta}{dt} \right)$$

$$\frac{\partial \Theta}{\partial y} = \frac{d\Phi}{d\eta} \frac{\partial \eta}{\partial y} = \frac{d\Phi}{d\eta} \frac{1}{\delta} \quad \text{and} \quad \frac{\partial^2 \Theta}{\partial y^2} = \frac{d^2 \Phi}{d\eta^2} \frac{1}{\delta^2}$$

b. We now substitute the above derivatives into the heat conduction equation to get (after multiplication by δ^2)

$$-\frac{d\Phi}{d\eta} \left(\eta \delta \frac{d\delta}{dt} \right) = \alpha_0 \frac{d}{d\eta} \left(\frac{d\Phi}{d\eta} + \beta \Phi \frac{d\Phi}{d\eta} \right)$$

When this is integrated over η from 0 to 1, we get

$$-\int_0^1 \left[\frac{d}{d\eta} (\eta \Phi) - \Phi \right] d\eta \cdot \delta \frac{d\delta}{dt} = \alpha_0 \int_0^1 \frac{d}{d\eta} \left(\frac{d\Phi}{d\eta} + \beta \Phi \frac{d\Phi}{d\eta} \right) d\eta$$

Performing the integrations we finally end up with

$$-\eta \Phi \Big|_0^1 + \int_0^1 \Phi d\eta \cdot \delta \frac{d\delta}{dt} = \alpha_0 \left(\frac{d\Phi}{d\eta} + \beta \Phi \frac{d\Phi}{d\eta} \right) \Big|_0^1$$

which is equivalent to Eqs. 12C.2-3 to 5.

c. We now use $\Phi = 1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3$, which is a good choice for the approximate temperature profile, since it gives $\Phi(0) = 1$, $\Phi(1) = 0$, and $\Phi'(1) = 0$ in agreement with our intuition.

We then get M and N by substitution into Eqs. 12C.2-4 and 5:

$$M = \int_0^1 \Phi d\eta = \int_0^1 \left(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right) d\eta = \frac{3}{8}$$

$$\begin{aligned} N &= \left(\frac{d\Phi}{d\eta} + \beta\Phi \frac{d\Phi}{d\eta} \right) \Big|_0^1 = \left[\left(-\frac{3}{2} + \frac{3}{2}\eta^2 \right) + \beta \left(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3 \right) \left(-\frac{3}{2} + \frac{3}{2}\eta^2 \right) \right] \Big|_0^1 \\ &= \frac{3}{2}(1 + \beta) \end{aligned}$$

This leads to the differential equation for the boundary layer thickness:

$$\frac{3}{8}\delta \frac{d\delta}{dt} = \frac{3}{2}(1 + \beta)\alpha_0$$

which when integrated gives

$$\delta = \sqrt{8(1 + \beta)\alpha_0 t}$$

The time-dependent temperature distribution is then

$$\frac{T - T_0}{T_1 - T_0} = 1 - \frac{3}{8} \left(\frac{y}{\sqrt{8(1 + \beta)\alpha_0 t}} \right) + \frac{1}{2} \left(\frac{y}{\sqrt{8(1 + \beta)\alpha_0 t}} \right)^3$$

The heat flux at $y = 0$ is finally

$$q_y \Big|_{y=0} = -k \frac{\partial T}{\partial y} \Big|_{y=0} = -\frac{k(T_1 - T_0)}{\delta(t)} \frac{d\Phi}{d\eta} \Big|_{\eta=0} = \frac{3}{2} \frac{k(T_1 - T_0)}{\sqrt{8(1 + \beta)\alpha_0 t}}$$

in which k is given by Eq. 12C.2-1.

12C.3 Heat conduction with phase change (the Neumann-Stefan problem)

a. Using the definition of the dimensionless temperature differences in Eqs. 12C.3-1 and 2, we may write the heat conduction equations for the solid and liquid phases as follows:

$$\text{Solid: } \frac{\partial \Theta_s}{\partial t} = \alpha \frac{\partial^2 \Theta_s}{\partial z^2} \quad \text{Liquid: } \frac{\partial \Theta_L}{\partial t} = \alpha \frac{\partial^2 \Theta_L}{\partial z^2}$$

The initial and boundary conditions are:

$$\begin{aligned} \text{I. C.:} & \quad \text{at } t = 0, & \quad \Theta_L = 1 \\ \text{B. C. 1:} & \quad \text{at } z = 0, & \quad \Theta_s = 0 \\ \text{B. C. 2:} & \quad \text{at } z = \infty, & \quad \Theta_L = 1 \\ \text{B. C. 3:} & \quad \text{at } z = Z(t), & \quad \Theta_L = \Theta_s = \Theta_m \\ \text{B. C. 4:} & \quad \text{at } z = Z(t), & \quad k \frac{\partial \Theta_s}{\partial z} - k \frac{\partial \Theta_L}{\partial z} = \frac{\rho \Delta \hat{H}_f}{T_1 - T_0} \frac{dZ}{dt} \end{aligned}$$

b. The assumed forms for the solution are chosen because of their similarity to other one-dimensional heat-flow problems in a semi-infinite region:

$$\text{Solid: } \Theta_s = C_1 + C_2 \operatorname{erf} \frac{z}{\sqrt{4\alpha t}}$$

$$\begin{aligned} \text{Liquid: } \Theta_L &= C_3 + C_4 \operatorname{erf} \frac{z}{\sqrt{4\alpha t}} = (C_3 + C_4) - C_4 \left(1 - \operatorname{erf} \frac{z}{\sqrt{4\alpha t}} \right) \\ &= (C_3 + C_4) - C_4 \operatorname{erfc} \frac{z}{\sqrt{4\alpha t}} \end{aligned}$$

The last form for the liquid-phase temperature equation will turn out to be somewhat more convenient to use.

c. When we apply B. C. 1, we get $C_1 = 0$. Using B. C. 2, we get $C_3 + C_4 = 1$; we note that the initial condition is automatically satisfied. Then B. C. 3 gives

$$C_2 \operatorname{erf} \frac{Z(t)}{\sqrt{4\alpha t}} = 1 - C_4 \operatorname{erfc} \frac{Z(t)}{\sqrt{4\alpha t}} = \Theta_m$$

The only way that this equation can be satisfied, is if $Z(t) \propto \sqrt{t}$. We elect to write, then, $Z(t) = \lambda \sqrt{4\alpha t}$ where λ is a dimensionless constant that has yet to be determined.

Finally we apply B. C. 4 to get

$$kC_2 \frac{2}{\sqrt{\pi}} e^{-z^2/4\alpha t} \left(\frac{1}{\sqrt{4\alpha t}} \right) - k(-C_4) \frac{2}{\sqrt{\pi}} e^{-z^2/4\alpha t} \left(\frac{-1}{\sqrt{4\alpha t}} \right) \\ = \frac{\rho \Delta \hat{H}_f}{T_1 - T_0} \lambda \sqrt{4\alpha} \frac{1}{2\sqrt{t}}$$

or

$$ke^{-\lambda^2} (C_2 - C_4) = \frac{\rho \Delta \hat{H}_f \lambda \alpha \sqrt{\pi}}{T_1 - T_0} \quad (*)$$

Next we apply that part of B. C. 3 that deals with the dimensionless melting temperature:

$$C_2 \operatorname{erf} \lambda = \Theta_m \quad \text{and} \quad 1 - C_4 \operatorname{erfc} \lambda = \Theta_m$$

whence

$$C_2 - C_4 = \frac{\Theta_m}{\operatorname{erf} \lambda} - \frac{1 - \Theta_m}{\operatorname{erfc} \lambda} \quad (**)$$

Combining (*) and (**) and rearranging, we get

$$\frac{\Theta_m}{\operatorname{erf} \lambda} - \frac{1 - \Theta_m}{\operatorname{erfc} \lambda} = \sqrt{\pi} \Lambda \lambda e^{+\lambda^2}$$

This gives λ in terms of Λ and Θ_m . Then we get for the temperature

$$\text{Solid: } \Theta_s = \Theta_m \frac{\operatorname{erf}(z/4\alpha t)}{\operatorname{erf} \lambda}; \quad \text{Liquid: } \Theta_L = 1 - (1 - \Theta_m) \frac{\operatorname{erfc}(z/4\alpha t)}{\operatorname{erfc} \lambda}$$

Finally, we have $Z(t) = \lambda \sqrt{4\alpha t}$, where λ is now a known function of Λ and Θ_m .

12C.4 Viscous heating in oscillatory flow

a. The starting point for this problem is Eq. 4.1-44, and the development up to and including Eq. 4.1-50 can be taken over here. B.C. 1 is also valid, but now B.C. 2 must read: at $y = b$, $v^o = 0$. When Eq. 4.1-50 is solved for these boundary conditions, we get

$$\frac{v^o(\xi)}{v_0} = \frac{\sinh c(1-\xi)}{\sinh c}$$

where $c = \sqrt{i\omega b^2/\nu} = \sqrt{\omega b^2/2\nu}(1+i) \equiv a(1+i)$ and $\xi = x/b$. The above expression for $v^o(\xi)$ can now be substituted into Eq. 4.1-48 to get Eq. 12C.4-1.

b. Next we get the dissipation function, which for this problem is $\Phi_v = \mu(\partial v_z/\partial x)^2$. The derivative of the velocity is

$$\frac{\partial v_z}{\partial x} = \Re \left\{ \frac{dv^o}{dx} e^{i\omega t} \right\}$$

and its square is obtained by using the relation (which should be proven) $\Re\{u\}\Re\{v\} = \frac{1}{2}[\Re\{uv\} + \Re\{uv^*\}]$, where u and v are complex numbers and the asterisk indicates a complex conjugate:

$$\left(\frac{\partial v_z}{\partial x} \right)^2 = \frac{1}{2} \left[\Re \left\{ \left(\frac{dv^o}{dx} \right)^2 e^{2i\omega t} \right\} + \left(\frac{dv^o}{dx} \right) \left(\frac{dv^o}{dx} \right)^* \right]$$

Next we get the time average of the above function; the first term vanishes because of the exponential (which contains sine and cosine functions) and only the second term survives:

$$\overline{\left(\frac{\partial v_z}{\partial x} \right)^2} = \frac{1}{2b^2} \left(\frac{dv^o}{dx} \right) \left(\frac{dv^o}{dx} \right)^* \quad \text{where} \quad \frac{dv^o}{dx} = -\frac{v_0}{b} \frac{c \cosh c(1-\xi)}{\sinh c}$$

from part (a). Then the time averaged dissipation function is:

$$\begin{aligned}
\overline{\Phi}_v &= \left(\frac{\partial v_z}{\partial x} \right)^2 = \frac{1}{2} \left(\frac{v_0}{b} \right)^2 \left(\frac{c \cosh c(1-\xi)}{\sinh c} \right) \left(\frac{c \cosh c(1-\xi)}{\sinh c} \right)^2 \\
&= \frac{1}{2} \left(\frac{v_0}{b} \right)^2 \frac{(cc^*)(\cosh c(1-\xi))(\cosh c(1-\xi))^*}{(\sinh c)(\sinh c)^*} \\
&= \frac{1}{2} \left(\frac{v_0}{b} \right)^2 \frac{[\cosh^2 a(1-\xi) \cos^2 a(1-\xi) + \sinh^2 a(1-\xi) \sin^2 a(1-\xi)]}{\sinh^2 a \cos^2 a + \cosh^2 a \sin^2 a}
\end{aligned}$$

To get the third line of the above we have used the identity

$$\cosh a(1+i)(1-\xi) = \cosh a(1-\xi) \cos a(1-\xi) + i \sinh a(1-\xi) \sin a(1-\xi)$$

which should be verified. Then, finally, use $\sin^2 \theta + \cos^2 \theta = 1$ and $\cosh^2 \theta - \sinh^2 \theta = 1$ to simplify the above expression for $\overline{\Phi}_v$, thus:

$$\overline{\Phi}_v = a^2 \left(\frac{v_0}{b} \right)^2 \left(\frac{\cos^2 a(1-\xi) + \sinh^2 a(1-\xi)}{\sin^2 a + \sinh^2 a} \right) \rightarrow a^2 \left(\frac{v_0}{b} \right)^2 e^{-2a\xi}$$

the last expression being a limiting expression for very large frequencies.

c. To get the time averaged temperature distribution, we use Eq. 12C.4-4, thus

$$\frac{d^2 \overline{T}}{d\xi^2} = -\frac{\mu b^2}{k} a^2 \left(\frac{v_0}{b} \right)^2 \left(\frac{\cos^2 a(1-\xi) + \sinh^2 a(1-\xi)}{\sin^2 a + \sinh^2 a} \right)$$

Then introducing a new variable $\eta = 1 - \xi$ we write

$$\frac{d^2 \overline{T}}{d(a\eta)^2} = -\frac{\mu v_0^2}{k} \left(\frac{\cos^2 a\eta + \sinh^2 a\eta}{\sin^2 a + \sinh^2 a} \right)$$

Integration once then gives

$$\frac{d\overline{T}}{d(a\eta)} = -\frac{\mu v_0^2}{4k} \left(\frac{\sin 2a\eta + \sinh 2a\eta}{\sin^2 a + \sinh^2 a} \right) + C_1$$

and a further integration then yields

$$\bar{T} = -\frac{\mu v_0^2}{4k} \left(\frac{\sin^2 a\eta + \sinh^2 a\eta}{\sin^2 a + \sinh^2 a} \right) + C_1 a\eta + C_2$$

The constants of integration may then be determined, so that we get finally:

$$\bar{T} - T_0 = \frac{\mu v_0^2}{4k} \left[(1 - \xi) - \frac{\sin^2 a(1 - \xi) + \sinh^2 a(1 - \xi)}{\sin^2 a + \sinh^2 a} \right]$$

For the high-frequency limit, we get

$$\bar{T} - T_0 = \frac{\mu v_0^2}{4k} \left[(1 - e^{-2a\xi}) - \xi(1 - e^{-2a}) \right]$$

(*Note:* The solution given here for finding the dissipation function does not use Eq. 12C.4-1 and is hence somewhat easier than the method suggested in the text. It does, however, require somewhat more familiarity with doing manipulations with complex variables.)

12C.5 Solar Heat Penetration

The basis for this problem was Eq. 5-40, M. N. Özışik, Heat Conduction, 2nd Ed., Wiley 1993 --- p.206. The temperature profile is re-expressed as

$$T^* \equiv \frac{T(y,t) - T(\infty)}{T_{\max}(0) - T(\infty)}$$

and should have been written as

$$T^* = \exp[-\sqrt{y^2 \omega / 2\alpha}] \cdot \cos[\omega t - \sqrt{y^2 \omega / 2\alpha}] - \frac{2}{\sqrt{\pi}} \int_0^{y/\sqrt{4\alpha}} \exp[-\eta^2] \cdot \cos[\omega t - y^2 \omega / 4\alpha \eta^2] d\eta$$

The thermal properties are taken from (Rohsenow, Hartnett and Cho, Handbook of Heat Transfer, McGraw-Hill, Third Ed., p.2.68, 1998):

Density, kg/m ³	Thermal conductivity, W/m,K	Specific heat, J/kg,K
1515	0.027	800

It follows that

$$\alpha = \frac{0.027}{800 \cdot 1515} \cdot \frac{m^2 W \cdot 3600 \text{ joules}}{\text{joule} - W - hr} = 8.02 \cdot 10^{-5} \frac{m^2}{hr}$$

a) For the assumption of a sinusoidal input with a 24 hour period

$$T(0,t) = a \sin(n\omega t)$$

Show that the mean temperature, in the absence of day-to-day variation is just a_0 , and for a periodicity of 24 hrs

$$\omega t = \pi t / 12 \text{ hrs}; \omega = \pi / 12 \text{ hrs}$$

The amplitude of the fluctuations at depth y relative to those at the surface, "A_{rel}" is just

$$A_{rel} = \exp\left(-y\sqrt{2\omega/\alpha}\right) \exp(-40.4 \cdot y/m)$$

Then at a depth of 10 cm the relative amplitude

$$A_{rel} = \exp(-4.04) = 0.0176$$

as required.

b) The transient term is difficult to evaluate, but this is of no practical importance. The periodically steady oscillations are about the initial temperature of zero, and therefore all transients must be of lesser magnitude than these periodic excursions. Since the excursions are already negligible there is no need to examine this problem further.

c) Only the slowest component of the oscillations need be considered as the others damp out even faster.

12C.6 Heat transfer in a falling non-Newtonian film

For the non-Newtonian falling film problem, the velocity distribution has been found in Problem 8B.1:

$$v_z = \left(\frac{\rho g \delta}{m} \right)^{1/n} \frac{\delta}{(1/n)+1} \left[1 - \left(\frac{x}{\delta} \right)^{(1/n)+1} \right] = v_{z,\max} \left[1 - \left(\frac{x}{\delta} \right)^{(1/n)+1} \right]$$

This can be rewritten in terms of the coordinate y , which is the distance from the solid surface

$$v_z = v_{z,\max} \left[1 - \left(1 - \frac{y}{\delta} \right)^{(1/n)+1} \right]$$

When this expression is expanded in a Taylor series, we get

$$v_z = v_{z,\max} \left[1 - 1 + \left(\frac{1}{n} + 1 \right) \left(\frac{y}{\delta} \right) - \frac{1}{2!} \left(\frac{1}{n} + 1 \right) \left(\frac{1}{n} \right) \left(\frac{y}{\delta} \right)^2 + \dots \right]$$

For positions very close to the wall, this can be approximated by

$$v_z = v_{z,\max} \left(\frac{1}{n} + 1 \right) \left(\frac{y}{\delta} \right)$$

Inserting the expression for the maximum velocity, we get

$$v_z = \left(\frac{\rho g \delta}{m} \right)^{1/n} y$$

which simplifies to Eq. 12B.4-1 for a Newtonian fluid.

Therefore the solution in Problem 12B.4-1 may be taken over for a power-law fluid by replacing $\beta = \mu k / \rho^2 \hat{C}_p g \delta = (k / \rho \hat{C}_p) (\mu / \rho g \delta)$ (just below Eq. 12B.4-3) by $\beta = (k / \rho \hat{C}_p) (m / \rho g \delta)^{1/n}$. The quantity β enters into Eq. 12B.4-8 via the variable η , and explicitly in Eq. 12B.4-9.

12D.1 Unsteady-state heating of a slab (Laplace-transform method)

a. We take the Laplace transform of Eq. 12.1-14 along with the initial conditions and boundary conditions given in Eqs. 12.1-15 and 16. This gives:

$$p\bar{\Theta} - 1 = \frac{d^2\bar{\Theta}}{d\eta^2} \quad \text{with} \quad \bar{\Theta}(\pm 1) = 0$$

This second-order, nonhomogeneous, ordinary differential equation can be solved by getting the complementary function (from Eq. C.1-4a) and the particular integral (by trial and error); this gives

$$\bar{\Theta} = C_1 \cosh \sqrt{p}\eta + C_2 \sinh \sqrt{p}\eta + \frac{1}{p}$$

When the constants are evaluated using the boundary conditions, we find that

$$\bar{\Theta} = \frac{1}{p} - \frac{\cosh \sqrt{p}\eta}{p \cosh \sqrt{p}}$$

Taking the inverse Laplace transform, we get

$$\begin{aligned} \Theta &\equiv \frac{T_1 - T}{T_1 - T_0} = 1 - \mathfrak{L}^{-1} \left\{ \frac{\cosh \sqrt{p}\eta}{p \cosh \sqrt{p}} \right\} \\ &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n \cos \left[\left(n + \frac{1}{2} \right) \pi \eta \right]}{\left(n + \frac{1}{2} \right) \pi} \exp \left[\left(n + \frac{1}{2} \right)^2 \pi^2 \tau \right] \end{aligned}$$

The inversion was performed by using Eq. (40) on p. 259 of Erdély, W. Magnus, F. Oberhettinger, and F. G. Tricomi *Tables of Integral Transforms*, McGraw-Hill, New York (1954), with ω set equal to zero. Unfortunately, this formula has two misprints in it: the x in the denominator of the first term should be l , and the factor $2p$ in front of the summation sign should be 2π . The inversion can also be performed by using the complex inversion theorem, as explained in Carslaw and Jaeger (see reference at the bottom of p. 403).

b. The expression for the transformed function $\bar{\Theta}$ (in (a)) can also be written in terms of exponentials. Then we expand part of the denominator in a binomial expansion to get:

$$\begin{aligned}\bar{\Theta} &= \frac{1}{p} - \frac{e^{\sqrt{p}\eta} + e^{-\sqrt{p}\eta}}{p(e^{\sqrt{p}} + e^{-\sqrt{p}})} = \frac{1}{p} - \frac{1}{p} \left(e^{-\sqrt{p}(1-\eta)} + e^{-\sqrt{p}(1+\eta)} \right) \sum_{n=0}^{\infty} (-1)^n e^{-2n\sqrt{p}} \\ &= \frac{1}{p} - \frac{1}{p} \sum_{n=0}^{\infty} (-1)^n e^{-\sqrt{p}(2n+1-\eta)} - \frac{1}{p} \sum_{n=0}^{\infty} (-1)^n e^{-\sqrt{p}(2n+1+\eta)}\end{aligned}$$

The inverse Laplace transform is then

$$\Theta \equiv \frac{T_1 - T}{T_1 - T_0} = 1 - \sum_{n=0}^{\infty} (-1)^n \operatorname{erfc} \frac{2n+1-\eta}{\sqrt{4\tau}} - \sum_{n=0}^{\infty} (-1)^n \operatorname{erfc} \frac{2n+1+\eta}{\sqrt{4\tau}}$$

This can be rewritten in terms of the dimensionless temperature of Example 12.1-1 thus:

$$\Phi \equiv \frac{T - T_0}{T_1 - T_0} = \sum_{n=0}^{\infty} (-1)^n \operatorname{erfc} \frac{2n+1-\eta}{\sqrt{4\tau}} + \sum_{n=0}^{\infty} (-1)^n \operatorname{erfc} \frac{2n+1+\eta}{\sqrt{4\tau}}$$

Here, the dimensionless temperature Φ has been introduced to avoid confusion with the dimensionless temperature Θ used in Example 12.1-2.

c. To compare the results of Examples 12.1-1 and 2, we replace the position variable y of Example 12.1-1 by x (so that it will not be confused with the y of Example 12.1-2). Then the first few terms of the final result in (b) are, written in terms of $x = b - y$, are:

$$\begin{aligned}\frac{T - T_0}{T_1 - T_0} &= \operatorname{erfc} \frac{1 - [1 - (x/b)]}{\sqrt{4\alpha t/b^2}} - \operatorname{erfc} \frac{3 - [1 - (x/b)]}{\sqrt{4\alpha t/b^2}} + \dots \\ &+ \operatorname{erfc} \frac{1 + [1 - (x/b)]}{\sqrt{4\alpha t/b^2}} - \operatorname{erfc} \frac{3 + [1 - (x/b)]}{\sqrt{4\alpha t/b^2}} + \dots = \operatorname{erfc} \frac{x}{\sqrt{4\alpha t}} + \dots\end{aligned}$$

the subsequent terms being smaller.

12D.2 The Graetz-Nusselt Problem

a. The partial differential equation to be solved is

$$\rho \hat{C}_p v_{z,\max} \left[1 - \left(\frac{r}{R} \right)^2 \right] \frac{\partial T}{\partial z} = k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \right]$$

When this is put in nondimensional form, using the dimensionless quantities defined in Eq. 12D.2-1, the following equation is obtained for the temperature distribution $\Theta(\xi, \zeta)$:

$$\phi(\xi) \frac{\partial \Theta}{\partial \zeta} = \frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial \Theta}{\partial \xi} \right) \text{ with } \Theta(\xi, 0) = 1, \Theta(1, \zeta) = 0, \text{ and } \left. \frac{\partial \Theta}{\partial \xi} \right|_{\xi=0} = 0$$

We now try the method of separation of variables with the dimensionless temperature given by $\Theta(\xi, \zeta) = X(\xi)Z(\zeta)$. This gives, after division by XZ

$$\frac{1}{Z} \frac{dZ}{d\zeta} = \frac{1}{X\phi} \frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{dX}{d\xi} \right)$$

Since the left side is a function of ζ alone, and the right side is a function of ξ alone, both sides have to equal a constant. We choose this constant to be $-\beta^2$. Hence we get two ordinary differential equations:

$$\frac{dZ}{d\zeta} = -\beta^2 Z \quad \text{or} \quad Z(\zeta) \propto \exp(-\beta^2 \zeta)$$

and

$$\frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{dX}{d\xi} \right) + \beta^2 \phi X = 0 \quad \text{with } X(1) = 0 \text{ and } X'(0) = 0$$

This last equation (with boundary conditions) is a boundary-value problem of the Sturm-Liouville type. Therefore, we know that there will be an infinite number of values of eigenvalues β_i^2 and an infinite

number of eigenfunctions $X_i(\xi)$. We also know that the eigenfunctions are orthogonal on the range (0,1) with respect to the weight function $\phi\xi$. Hence the solution must have the form:

$$\Theta(\xi, \zeta) = \sum_{i=1}^{\infty} A_i X_i(\xi) \exp(-\beta_i^2 \zeta) \quad \text{with } A_i = \frac{\int_0^1 X_i \phi \xi d\xi}{\int_0^1 X_i^2 \xi d\xi}$$

The expression for A_i is obtained by using the requirement that $\Theta(\xi, 0) = 1$, and making use of the orthogonality relations for the eigenfunctions.

b. We write for the Newtonian fluid

$$\phi(\xi) = v_z / \langle v_z \rangle = v_{z,\max} (1 - \xi^2) / \langle v_z \rangle = 2(1 - \xi^2)$$

so that Eq. 12D.2-3 becomes:

$$\frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{dX_i}{d\xi} \right) + 2\beta_i^2 (1 - \xi^2) X_i = 0 \quad \text{with } X(1) = 0 \text{ and } X'(0) = 0$$

This second-order equation will have two solutions, but one of them will become infinite at $\xi = 0$ and is therefore unacceptable. The remaining solution we write as a power series

$$X_i(\xi) = \sum_{j=0}^{\infty} b_{ij} \xi^j \quad (*)$$

When this is substituted into the differential equation, we get

$$\sum_{j=0}^{\infty} j^2 b_{ij} \xi^{j-2} + 2\beta_i^2 (1 - \xi^2) \sum_{j=0}^{\infty} b_{ij} \xi^j = 0$$

By collecting equal powers of ξ we get the recursion relation for the coefficients:

$$b_{ij} = (2\beta_i^2 / j^2) (b_{i,j-4} - b_{i,j-2}) \quad (*)$$

The b_{i0} are arbitrarily chosen to be unity, and the higher coefficients are obtained from the equation immediately above. The values of β_i^2 are then obtained from the two equations marked with (*), along with the boundary condition at $\xi = 1$, which requires that $\sum_{j=0}^{\infty} b_{ij} = 0$.

This involves solving an infinite order algebraic equation by trial and error--clearly a tedious process. Methods are, however, available to get the eigenvalues β_i^2 for small and large values of i . For small values there is the method of Stodola and Vianello, and for large values there is the WKB (Wentzel-Kramers-Brioullin) method. The following eigenvalues have been obtained:

i	1	2	3	4	5	6
$2\beta_i^2$	7.314	44.61	113.9	215.1	348.4	513.8

See B. C. Lyche and R. B. Bird, *Chem. Eng. Sci.*, 6, 35-41 (1956), for the first three values both for Newtonian and power-law fluids; the higher values are obtained from the WKB method as $2\beta_i^2 = (4i - \frac{4}{3})^2$. Further discussions and literature references for non-Newtonian flow may be found in the 1st edition (Chapter 5) and the 2nd edition (Chapter 6) of R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Vol. 1, *Fluid Mechanics* (published by Wiley-Interscience).

12D.3 The Graetz-Nusselt problem (asymptotic solution for large z)

a. Using the definition of the dimensionless temperature as given in Problem 12D.2-2, we can write

$$q_0 = q_r|_{r=R} = -k \frac{\partial T}{\partial r} \Big|_{r=R} = -\frac{k(T_1 - T_0)}{R} \frac{\partial \Theta}{\partial \xi} \Big|_{\xi=1} = -\frac{k(T_b - T_0)}{R\Theta_b} \frac{\partial \Theta}{\partial \xi} \Big|_{\xi=1}$$

b. For very large z ,

$$\frac{\partial \Theta}{\partial \xi} \Big|_{\xi=1} = \sum_{i=1}^{\infty} A_i \frac{dX_i}{d\xi} \Big|_{\xi=1} \exp(-\beta_i^2 \zeta) \approx A_1 \frac{dX_1}{d\xi} \Big|_{\xi=1} \exp(-\beta_1^2 \zeta)$$

Furthermore, from the definition of the bulk temperature in Eq. 10.8-33 we find using Eqs. 12D.2-2 and 3

$$\begin{aligned} \Theta_b &= \frac{\int_0^1 \phi \Theta \xi d\xi}{\int_0^1 \phi \xi d\xi} = 2 \int_0^1 \phi \Theta \xi d\xi = 2 \sum_{i=1}^{\infty} A_i \exp(-\beta_i^2 \zeta) \int_0^1 \phi X_i \xi d\xi \\ &= -2 \sum_{i=1}^{\infty} A_i \exp(-\beta_i^2 \zeta) \int_0^1 \frac{1}{\beta_i^2} \frac{d}{d\xi} \left(\xi \frac{dX_i}{d\xi} \right) d\xi \\ &= -2 \sum_{i=1}^{\infty} A_i \exp(-\beta_i^2 \zeta) \frac{1}{\beta_i^2} \left(\xi \frac{dX_i}{d\xi} \right) \Big|_0^1 \\ &= -2 \sum_{i=1}^{\infty} A_i \exp(-\beta_i^2 \zeta) \frac{1}{\beta_i^2} \frac{dX_i}{d\xi} \Big|_{\xi=1} \approx -2A_1 \exp(-\beta_1^2 \zeta) \frac{1}{\beta_1^2} \frac{dX_1}{d\xi} \Big|_{\xi=1} \end{aligned}$$

the very last expression being valid for large z . When these results are substituted into the result in (a), we get

$$q_0 = -\frac{k}{R} (T_b - T_0) \cdot \left(-\frac{\beta_1^2}{2} \right) = \beta_1^2 \frac{k(T_b - T_0)}{D}$$

This is the same as Eq. 12D.3-3 in the text. (see also p. 217 of R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Vol. 1, *Fluid Dynamics*, Wiley-Interscience, New York, 2nd edition (1987)).

12D.4 The Graetz-Nusselt problem (asymptotic expression for small z)

a. When Eq. 10.8-12 is written in terms of the dimensionless variables $\Theta = (T_1 - T)/(T_1 - T_0)$, $\xi = r/R$, and $\zeta = z/R$, it becomes

$$v_{z,\max} (1 - \xi^2) \frac{\partial \Theta}{\partial \zeta} = \frac{\alpha}{R} \frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial \Theta}{\partial \xi} \right)$$

Next we introduce the three assumptions used for small z in Example 12.2-2, as well as the dimensionless quantities in Eq. 12.4-1. This gives for the partial differential equation for $\Theta(\sigma, \xi)$ the following problem statement:

$$N\sigma \frac{\partial \Theta}{\partial \zeta} = \frac{\partial^2 \Theta}{\partial \sigma^2} \quad \text{with } \Theta(\sigma, 0) = 0, \Theta(0, \zeta) = 1, \Theta(\infty, \zeta) = 0$$

We use the method of combination of independent variables, by introducing the new independent variable: $\eta = (N\sigma^3/9\zeta)^{1/3}$. Then the transformation of the derivatives proceeds as follows:

$$\left(\frac{\partial \Theta}{\partial \zeta} \right)_\sigma = \frac{d\Theta}{d\eta} \left(\frac{\partial \eta}{\partial \zeta} \right)_\sigma = \frac{d\Theta}{d\eta} \left(\frac{N\sigma^3}{9} \right)^{1/3} \left(-\frac{1}{3} \right) \zeta^{-4/3} = -\frac{\eta}{3\zeta} \frac{d\Theta}{d\eta}$$

$$\left(\frac{\partial \Theta}{\partial \sigma} \right)_\zeta = \frac{d\Theta}{d\eta} \left(\frac{\partial \eta}{\partial \sigma} \right)_\zeta = \frac{d\Theta}{d\eta} \left(\frac{N}{9\zeta} \right)^{1/3} = \frac{d\Theta}{d\eta} \frac{\eta}{\sigma} \quad \text{and} \quad \left(\frac{\partial^2 \Theta}{\partial \sigma^2} \right)_\zeta = \frac{d^2 \Theta}{d\eta^2} \left(\frac{\eta}{\sigma} \right)^2$$

When these substitutions are made into the partial differential equation in dimensionless form, we get (after some simplification) the following ordinary differential equation and boundary conditions for $\Theta(\eta)$

$$\frac{d^2 \Theta}{d\eta^2} + 3\eta^2 \frac{d\Theta}{d\eta} = 0 \quad \text{with} \quad \Theta(0) = 1 \quad \text{and} \quad \Theta(\infty) = 0$$

} D

To solve this equation, we set $\Phi(\eta) = d\Theta/d\eta$, and obtain a separable first-order differential equation for Φ :

$$\frac{d\Phi}{d\eta} = -3\eta^2\Phi \quad \text{with solution} \quad \frac{d\Theta}{d\eta} = \Phi = C_1 \exp(-\eta^3)$$

This first-order separable equation can be integrated to give

$$\Theta = C_1 \int_0^\eta \exp(-\bar{\eta}^3) d\bar{\eta} + C_2$$

The lower limit of the integral has been chosen arbitrarily. Application of the boundary conditions enables us to determine the constants of integration. The final solution is then

$$\Theta = 1 - \frac{\int_0^\eta \exp(-\bar{\eta}^3) d\bar{\eta}}{\int_0^\infty \exp(-\bar{\eta}^3) d\bar{\eta}} = \frac{\int_\eta^\infty \exp(-\bar{\eta}^3) d\bar{\eta}}{\int_0^\infty \exp(-\bar{\eta}^3) d\bar{\eta}} = \frac{1}{\Gamma(\frac{4}{3})} \int_\eta^\infty \exp(-\bar{\eta}^3) d\bar{\eta}$$

where Eq. C.4-3 has been used to write the denominator integral in terms of a gamma function. The integral in the numerator cannot be evaluated analytically.

b. To get the wall heat flux, we use Fourier's law of heat conduction:

$$\begin{aligned} q_r|_{r=R} &= -k \frac{\partial T}{\partial r} \Big|_{r=R} = +\frac{k}{R} (T_1 - T_0) \frac{\partial \Theta}{\partial \xi} \Big|_{\xi=1} = -\frac{k}{R} (T_1 - T_0) \frac{\partial \Theta}{\partial \sigma} \Big|_{\sigma=0} \\ &= -\frac{k}{R} (T_1 - T_0) \left(\frac{N}{9\zeta} \right)^{1/3} \frac{\partial \Theta}{\partial \eta} \Big|_{\eta=0} \end{aligned}$$

Into this expression we substitute the final result of (a), to obtain (with the help of the Leibniz formula for differentiating an integral, given in Eq. C.3-2)

$$q_r|_{r=R} = -\frac{k}{R} (T_1 - T_0) \left(\frac{N}{9\zeta} \right)^{1/3} \left(-\frac{1}{\Gamma(\frac{4}{3})} \exp(-\eta^3) \Big|_{\eta=0} \right)$$

$$\begin{aligned}
&= +\frac{k}{R}(T_1 - T_0)\frac{1}{9^{1/3}\Gamma(\frac{4}{3})}\left(\frac{4\langle v_z \rangle R}{\alpha} \cdot \frac{R}{z}\right)^{1/3} \\
&= +\frac{k}{R}(T_1 - T_0)\frac{1}{9^{1/3}\Gamma(\frac{4}{3})}\left(\frac{D\langle v_z \rangle \rho}{\mu} \cdot \frac{\hat{C}_p \mu}{k} \cdot \frac{D}{z}\right)^{1/3} \\
&= \frac{k}{R}(T_1 - T_0)\frac{1}{9^{1/3}\Gamma(\frac{4}{3})}\left(\text{RePr} \frac{D}{z}\right)^{1/3}
\end{aligned}$$

c. In (a) we can write the velocity profile for the Newtonian fluid (or any generalized Newtonian fluid) as

$$v_z = \langle v_z \rangle \phi(\xi) \quad \text{with} \quad \phi(1) = 0 \quad (\text{no-slip condition})$$

That is, for the Newtonian fluid $\phi(\xi) = 2(1 - \xi^2)$. Then when we switched to the dimensionless coordinate σ measured from the wall, the velocity profile can be written as

$$v_z = \langle v_z \rangle \psi(\sigma) \quad \text{with} \quad \psi(0) = 0 \quad (\text{no-slip condition})$$

Then, the velocity profile in the immediate vicinity of the wall is

$$v_z = \langle v_z \rangle \psi'(0)\sigma + \dots = -\langle v_z \rangle \phi'(1)\sigma + \dots$$

and, for the Newtonian fluid,

$$v_z = -\langle v_z \rangle \left[\left(\frac{d}{d\xi} 2(1 - \xi^2) \right) \Big|_{\xi=1} \right] \sigma + \dots = 4\langle v_z \rangle \sigma + \dots$$

Hence, the above solution in (a) can be modified for the generalized Newtonian fluid by replacing $N = 4\langle v_z \rangle R/\alpha$ by

$$N = \left(-\frac{d\phi}{d\xi} \Big|_{\xi=1} \right) \frac{\langle v_z \rangle R}{\alpha}$$

12D.5 The Graetz problem for flow between parallel plates

a. Let the flow take place in the positive z direction between plates at $x = \pm B$. Then the partial differential boundary equation for $T(x, z)$ and the boundary conditions are:

$$\rho \hat{C}_p v_z(x) \frac{\partial T}{\partial z} = k \frac{\partial^2 T}{\partial x^2} \quad \text{with} \quad T(x, 0) = T_1, \quad T(\pm B, z) = T_0$$

We can, if we wish, replace the boundary condition at $x = -B$ by a boundary condition $\partial T / \partial x = 0$ at $x = 0$, because of the fact that we know that the distribution of temperature about the center plane will be symmetric. Then we solve the problem for positive x only, knowing that the full problem must be symmetric about the plane $x = 0$. We now introduce the dimensionless variables

$$\Theta(\xi, \zeta) = \frac{T - T_0}{T_1 - T_0} \quad \phi(\xi) = \frac{v_z(x)}{\langle v_z \rangle} \quad \xi = \frac{x}{B} \quad \zeta = \frac{\alpha z}{\langle v_z \rangle B^2}$$

Then the problem may be reformulated as

$$\phi(\xi) \frac{\partial \Theta}{\partial \zeta} = \frac{\partial^2 \Theta}{\partial \xi^2} \quad \text{with} \quad \Theta(\xi, 0) = 1, \quad \Theta(1, \zeta) = 0, \quad \left. \frac{\partial \Theta}{\partial \xi} \right|_{\xi=0} = 0$$

We anticipate that the method of separation of variables will be appropriate, and write $\Theta(\xi, \zeta) = X(\xi)Z(\zeta)$. When this is substituted into the equation above, we get (after division by XZ)

$$\frac{1}{Z} \frac{dZ}{d\zeta} = \frac{1}{X\phi} \frac{d^2 X}{d\xi^2} = -\beta^2$$

where we have introduced the separation constant $-\beta^2$. Hence we get two ordinary differential equations

$$\frac{dZ}{d\zeta} = -\beta^2 Z \quad \text{whence} \quad Z(\zeta) \propto \exp(-\beta^2 \zeta)$$

$$\frac{d^2 X}{d\xi^2} + \beta^2 \phi X = 0 \quad \text{with } X(1) = 0 \text{ and } X'(0) = 0$$

Since this last equation with its boundary conditions is a boundary-value problem of the Sturm-Liouville type, there will be an infinite number of values of the eigenvalues β_i^2 and an infinite number of eigenfunctions $X_i(\xi)$. Furthermore, it is known that the eigenfunctions are orthogonal on the range (0,1) with respect to the weight function $\phi(\xi)$. Therefore, the dimensionless temperature profiles must have the form:

$$\Theta(\xi, \zeta) = \sum_{i=1}^{\infty} A_i X_i(\xi) \exp(-\beta_i^2 \zeta) \quad (*) \quad \text{with } A_i = \frac{\int_0^1 X_i(\xi) \phi(\xi) d\xi}{\int_0^1 X_i^2(\xi) d\xi}$$

The expression for the A_i is obtained by using the requirement that $\Theta(\xi, 0) = 1$ and using the orthogonality relations for the eigenfunctions $X_i(\xi)$.

It remains to find the eigenfunctions $X_i(\xi)$ and the eigenvalues β_i^2 . This has to be done by solving the equation for $X_i(\xi)$ by a power-series technique. It is a tedious process. The results are given in the *Handbook of Heat Transfer*, by Rohsenow, Hartnett, and Cho, cited in a footnote in Table 12D.2 on p. 404.

b. In the limit of very large ζ , only one term in the summation in (*) is needed, that is, we need only $X_1(\xi)$ and β_1^2 . To get the heat flux into the wall at large distances downstream, we need to find

$$q_0 = +k \left. \frac{\partial T}{\partial x} \right|_{x=+B} = \frac{k(T_b - T_0)}{B} \frac{(\partial \Theta / \partial \xi)|_{\xi=1}}{\Theta_b} \quad \text{where} \quad \Theta_b = \frac{T_b - T_0}{T_1 - T_0}$$

and T_b is the bulk fluid temperature.

For very large ζ ,

$$\left. \frac{\partial \Theta}{\partial \xi} \right|_{\xi=1} = \sum_{i=1}^{\infty} A_i \left. \frac{dX_i}{d\xi} \right|_{\xi=1} \exp(-\beta_i^2 \zeta) \approx A_1 \left. \frac{dX_1}{d\xi} \right|_{\xi=1} \exp(-\beta_1^2 \zeta)$$

The bulk temperature is, for very large ζ obtained as follows:

$$\begin{aligned} \Theta_b(\zeta) &= \frac{\int_0^1 \phi \Theta d\xi}{\int_0^1 \phi d\xi} = \sum_{i=1}^{\infty} A_i \exp(-\beta_i^2 \zeta) \int_0^1 \phi X_i d\xi \\ &= - \sum_{i=1}^{\infty} A_i \exp(-\beta_i^2 \zeta) \int_0^1 \frac{1}{\beta_i^2} \frac{d^2 X_i}{d\xi^2} d\xi \\ &= - \sum_{i=1}^{\infty} A_i \exp(-\beta_i^2 \zeta) \left. \frac{1}{\beta_i^2} \frac{dX_i}{d\xi} \right|_0^1 \\ &= - \sum_{i=1}^{\infty} A_i \exp(-\beta_i^2 \zeta) \left. \frac{1}{\beta_i^2} \frac{dX_i}{d\xi} \right|_{\xi=1} \approx -A_1 \exp(-\beta_1^2 \zeta) \left. \frac{1}{\beta_1^2} \frac{dX_1}{d\xi} \right|_{\xi=1} \end{aligned}$$

Substituting the last two results into the expression for the heat flux, we find

$$q_0 = \frac{k(T_0 - T_b)}{B} \beta_1^2$$

In terms of the Nusselt number, this result may be written as

$$\text{Nu} = \frac{4hB}{k} = \frac{4q_0 B}{k(T_0 - T_b)} = 4\beta_1^2 = 7.541$$

which is the result cited in Eq. (F) in Table 14.2-2.

c. For the limiting case of very small values of z , we start with the same equation used in part (a) for $T(x, z)$:

$$v_z(x) \frac{\partial T}{\partial z} = \alpha \frac{\partial^2 T}{\partial x^2} \quad \text{with} \quad T(x, 0) = T_1, \quad T(\pm B, z) = T_0$$

Since we are concerned only with the temperature change in the immediate vicinity of the wall, it is convenient to change to a new coordinate $y = B - x$ and to expand the velocity in powers of y thus:

$$v_z = v_{z,\max} \left[1 - \left(\frac{x}{B} \right)^2 \right] = v_{z,\max} \left[1 - \left(1 - \frac{y}{B} \right)^2 \right] = v_{z,\max} \left[2 \left(\frac{y}{B} \right) - \left(\frac{y}{B} \right)^2 \right]$$

$$\approx 2v_{z,\max} \frac{y}{B} = 3\langle v_z \rangle \frac{y}{B}$$

where we have used the result from 2B.3(b); this truncated expression is valid in the vicinity of the wall for small values of z . The energy equation for $T(y, z)$ can then be written as

$$3\langle v_z \rangle \frac{y}{B} \frac{\partial T}{\partial z} = \alpha \frac{\partial^2 T}{\partial y^2} \quad \text{with } T(y, 0) = T_1, T(0, z) = T_0, T(\infty, z) = T_1$$

The last boundary condition is sufficiently good for small values of z , since taking the center plane as being at an infinite distance from the wall will not change the temperature distribution significantly. We now introduce the following dimensionless quantities:

$$\Theta = \frac{T_1 - T}{T_1 - T_0} \quad \eta = \frac{y}{B} \quad \zeta = \frac{z}{B} \quad N = \frac{3\langle v_z \rangle B}{\alpha}$$

Then the differential equation for $\Theta(\eta, \zeta)$ and the boundary conditions are:

$$N\eta \frac{\partial \Theta}{\partial \zeta} = \frac{\partial^2 \Theta}{\partial \eta^2} \quad \text{with } \Theta(\eta, 0) = 0, \Theta(0, \zeta) = 1, \Theta(\infty, \zeta) = 0$$

We now seek a solution by the method of combination of variables, using the combined variable $\chi = (N\eta^3/9\zeta)^{1/3}$. This is possible since the first and third boundary conditions both state that the dimensionless temperature is zero. We are then led to the ordinary differential equation (for the details, see the solution to Problem 12D.4)

$$\frac{d^2 \Theta}{d\chi^2} + 3\chi^2 \frac{d\Theta}{d\chi} = 0 \quad \text{with } \Theta(0) = 1 \text{ and } \Theta(\infty) = 0$$

for which the solution is

$$\Theta = \frac{1}{\Gamma\left(\frac{4}{3}\right)} \int_{\chi}^{\infty} \exp(-\bar{\chi}^3) d\bar{\chi}$$

To get the wall heat flux, we use Fourier's law

$$\begin{aligned} q_0 &= -k \frac{\partial T}{\partial y} \Big|_{y=0} = + \frac{k(T_1 - T_0)}{B} \frac{\partial \Theta}{\partial \eta} \Big|_{\eta=0} = + \frac{k(T_1 - T_0)}{B} \left(\frac{N}{9\zeta}\right)^{1/3} \frac{\partial \Theta}{\partial \chi} \Big|_{\chi=0} \\ &= + \frac{k(T_1 - T_0)}{B} \left(\frac{N}{9\zeta}\right)^{1/3} \frac{1}{\Gamma\left(\frac{4}{3}\right)} (-1) \exp(-\chi^3) \Big|_{\chi=0} \\ &= + \frac{k(T_0 - T_1)}{B} \left(\frac{N}{9\zeta}\right)^{1/3} \frac{1}{\Gamma\left(\frac{4}{3}\right)} \end{aligned}$$

The above result can be put in the form of a Nusselt number as follows:

$$\begin{aligned} \text{Nu} &= \frac{4hB}{k} = \frac{4q_0B}{k(T_0 - T_b)} \approx \frac{4q_0B}{k(T_0 - T_1)} \\ &= 4 \left(\frac{N}{9\zeta}\right)^{1/3} \frac{1}{\Gamma\left(\frac{4}{3}\right)} = 4 \left(\frac{3\langle v_z \rangle B}{9z/B}\right)^{1/3} \frac{1}{\Gamma\left(\frac{4}{3}\right)} \\ &= \frac{4}{(3)^{1/3} \Gamma\left(\frac{4}{3}\right)} \left(\frac{\langle v_z \rangle B}{z/B}\right)^{1/3} \end{aligned}$$

This is in agreement with Eq. (C) of Table 14.2-2. The replacement of the bulk temperature by the entrance temperature in the first line is appropriate for the entry region, inasmuch as heat has had the opportunity to penetrate only a negligible amount in that region.

12D.6 The constant wall heat flux problem for parallel plates

a. Application of §10.8 to parallel-plate system

For the laminar flow in the z direction between parallel plates at $x = \pm B$, the differential equation is

$$\rho \hat{C}_p v_{\max} \left[1 - \left(\frac{x}{B} \right)^2 \right] \frac{\partial T}{\partial z} = k \frac{\partial^2 T}{\partial x^2}$$

We now introduce the dimensionless variables

$$\xi = \frac{x}{B} \quad \zeta = \frac{kz}{\rho \hat{C}_p v_{\max} B^2} \quad \Theta(\xi, \zeta) = \frac{k(T - T_1)}{q_0 B}$$

When the differential equation is multiplied by B/q_0 , the above differential equation can be rewritten as

$$(1 - \xi^2) \frac{\partial \Theta}{\partial \zeta} = \frac{\partial^2 \Theta}{\partial \xi^2} \quad (**)$$

with the boundary conditions:

$$\begin{array}{lll} \text{B. C. 1:} & \text{at } \xi = 0, & \frac{\partial \Theta}{\partial \xi} = 0 \\ \text{B. C. 2:} & \text{at } \xi = +1, & \frac{\partial \Theta}{\partial \xi} = 1 \\ \text{B. C. 3:} & \text{at } \zeta = 0, & \Theta = 0 \end{array}$$

We now seek a solution valid for large downstream distance, and try a solution of the form

$$\Theta(\xi, \zeta) = C_0 \zeta + \Psi(\xi) \quad (*)$$

Since this function cannot satisfy B. C. 3, we replace the latter by B. C. 4:

$$\text{Condition 4:} \quad \text{at any plane } z, \quad 2zWq_0 = \int_0^W \int_{-B}^{+B} \rho \hat{C}_p (T - T_1) v_z dx dy$$

or, in dimensionless variables

$$\text{at any plane } \zeta, \quad \zeta = \int_0^1 \Theta(\xi, \zeta)(1 - \xi^2) d\xi$$

When the trial function in (*) is substituted into (**), we get

$$(1 - \xi^2)C_0 = \frac{d^2\Psi}{d\xi^2}$$

Integration of this equation gives

$$\frac{d\Psi}{d\xi} = C_0\left(\xi - \frac{1}{3}\xi^3\right) + C_1 \quad (***)$$

Further integration gives

$$\Psi = C_0\left(\frac{1}{2}\xi^2 - \frac{1}{12}\xi^4\right) + C_1\xi + C_2$$

so that

$$\Theta(\xi, \zeta) = C_0\zeta + C_0\left(\frac{1}{2}\xi^2 - \frac{1}{12}\xi^4\right) + C_1\xi + C_2$$

B. C. 1 leads to $C_1 = 0$. Furthermore, B. C. 2, with the help of (***), gives $C_0 = \frac{3}{2}$. Finally Condition 4 leads to:

$$\zeta = \int_0^1 \left[\frac{3}{2}\zeta + \frac{3}{2}\left(\frac{1}{2}\xi^2 - \frac{1}{12}\xi^4\right) + C_2 \right] (1 - \xi^2) d\xi$$

from which we get $C_2 = -\frac{39}{280}$. Therefore, finally

$$\Theta(\xi, \zeta) = \frac{3}{2}\zeta + \frac{3}{4}\left(\xi^2 - \frac{1}{6}\xi^4\right) - \frac{39}{280}$$

Then the bulk temperature as a function of downstream distance is

$$T_b - T_1 = \frac{\int_0^B [T(x, z) - T_1] v_z(x) dx}{\int_0^B v_z(x) dx} = \frac{q_0 B \int_0^1 \Theta(\xi, \zeta) (1 - \xi^2) d\xi}{k \int_0^1 (1 - \xi^2) d\xi} = \frac{q_0 B}{k} \left(\frac{3}{2} \zeta \right)$$

and the wall temperature as a function of downstream distance is

$$T_0 - T_1 = \frac{q_0 B}{k} \left(\frac{3}{2} \zeta + \frac{3}{4} \left(1 - \frac{1}{6} \right) - \frac{39}{280} \right) = \frac{q_0 B}{k} \left(\frac{3}{2} \zeta + \frac{17}{35} \right)$$

Therefore

$$T_0 - T_b = \frac{17}{35} \frac{q_0 B}{k} \quad \text{or} \quad \text{Nu} = \frac{4q_0 B}{k(T_0 - T_1)} = \frac{140}{17}$$

in agreement with Eq. (L) of Table 14.2-2 (note the definition of Nu in the table title!).

b. For very small values of z the velocity distribution the linear portion very near the wall is the only part of the distribution that is important; if we let y be the distance from the wall at $x = B$, we have

$$\begin{aligned} v_z &= v_{\max} \left[1 - \left(\frac{x}{B} \right)^2 \right] = v_{\max} \left[1 - \left(1 - \left(\frac{y}{B} \right) \right)^2 \right] \\ &= v_{\max} \left(2 \frac{y}{B} - \left(\frac{y}{B} \right)^2 \right) \approx 2 v_{\max} \frac{y}{B} = 3 \langle v_z \rangle \frac{y}{B} \equiv v_0 \frac{y}{B} \end{aligned}$$

which serves to define v_0 . Then the first equation in (a) becomes

$$v_0 \frac{y}{B} \frac{\partial T}{\partial z} = \alpha \frac{\partial^2 T}{\partial y^2} \quad \text{or} \quad v_0 \frac{1}{B} \frac{\partial T}{\partial z} = \alpha \frac{1}{y} \frac{\partial^2 T}{\partial y^2}$$

Aside from the replacement of R by B , these equations are identical to Eqs. 12.2-13 and 14, and they have the same boundary conditions. Therefore, the remainder of Example 12.2-2 is valid for the slit flow (with the replacement of R by B).

When $\eta = 0$, we also have $\chi = 0$, and Eq. 12.2-24 gives for the difference between the wall and bulk temperatures

$$\frac{T_0 - T_1}{(q_0 B/k)} = \frac{\sqrt[3]{9\lambda}}{\Gamma(\frac{2}{3})} = \frac{\sqrt[3]{9}}{\Gamma(\frac{2}{3})} \sqrt[3]{\frac{\alpha z}{v_0 B^2}} = \frac{\sqrt[3]{3}}{\Gamma(\frac{2}{3})} \sqrt[3]{\frac{\alpha z}{\langle v_z \rangle B^2}}$$

inasmuch as the bulk temperature is virtually the same as the entrance temperature, the heat having penetrated to only a very small distance into the fluid. Then the Nusselt number is

$$\text{Nu} = \frac{4hB}{k} = \frac{4q_0 B}{k(T_0 - T_1)} = \frac{4\Gamma(\frac{2}{3})}{\sqrt[3]{3}} \sqrt[3]{\frac{\langle v_z \rangle B^2}{\alpha z}}$$

in agreement with Eq. (H) of Table 14.2-2.

c. The slit analogs of Eqs. 10.8-12 and 19 are

$$\rho \hat{C}_p v_{z,\max} \left[1 - \left(\frac{x}{B} \right)^2 \right] \frac{\partial T}{\partial z} = k \frac{\partial^2 T}{\partial x^2} \quad \text{or} \quad (1 - \xi^2) \frac{\partial \Theta}{\partial \zeta} = \frac{\partial^2 \Theta}{\partial \xi^2}$$

In the second equation we have introduced the dimensionless variables

$$\Theta = \frac{T - T_1}{q_0 B/k} \quad \xi = \frac{x}{B} \quad \zeta = \frac{z}{\rho \hat{C}_p v_{z,\max} B^2 / k}$$

From §10.8 we know that the asymptotic solution for large downstream distances is

$$\Theta_\infty(\xi, \zeta) = \frac{3}{2} \zeta + \frac{3}{4} \left(\xi^2 - \frac{1}{6} \xi^4 \right) - \frac{39}{280}$$

The complete solution is assumed to be of the following form

$$\Theta(\xi, \zeta) = \Theta_\infty(\xi, \zeta) - \Theta_d(\xi, \zeta)$$

Substituting this into the partial differential equation above tells us that the time-decaying function $\Theta_d(\xi, \zeta)$ must satisfy the equation

$$(1 - \xi^2) \frac{\partial \Theta_d}{\partial \zeta} = \frac{\partial^2 \Theta_d}{\partial \xi^2}$$

with the following boundary conditions:

$$\text{B. C. 1:} \quad \text{at } \xi = 0, \quad \frac{\partial \Theta_d}{\partial \xi} = 0$$

$$\text{B. C. 2:} \quad \text{at } \xi = 1, \quad \frac{\partial \Theta_d}{\partial \xi} = 0$$

$$\text{B. C. 3:} \quad \text{at } \zeta = 0, \quad \Theta_d = \Theta_\infty(\xi, 0)$$

We try a solution by the method of separation of variables, letting $\Theta_d(\xi, \zeta) = X(\xi)Z(\zeta)$. The functions X and Z must then satisfy the following ordinary differential equations

$$\frac{dZ}{d\zeta} = -c^2 Z \quad \frac{d^2 X}{d\xi^2} + c^2(1 - \xi^2)X = 0 \quad (****)$$

where $-c^2$ is the separation constant. We then expect to have a complete solution of the form

$$\Theta(\xi, \zeta) = \Theta_\infty(\xi, 0) - \sum B_k \exp(-c_k^2 \zeta) X_k(\xi)$$

where

$$B_k = \frac{\int_0^1 \Theta_\infty(\xi, 0) [X_k(\xi)] (1 - \xi^2) d\xi}{\int_0^1 [X_k(\xi)]^2 (1 - \xi^2) d\xi}$$

Therefore, the problem is reduced to one of finding the eigenfunctions $X_k(\xi)$ of (****), and then getting the eigenvalues c_k , by applying the boundary condition at the wall $\xi = 1$.

The Nusselt number is shown in Fig. 14.2-1 over the entire range, including the limiting cases given in parts (a) and (b).

12D.7 Asymptotic solution for small z for laminar tube flow with constant heat flux

Exchange the order of integration in Eq. 12D.7-1, recognizing that the area of integration is a triangular area bounded by the lines $\bar{\chi} = \chi$, $\bar{\chi} = \infty$, and $\bar{\chi} = \infty$. Then the inner integration can be performed analytically:

$$\begin{aligned}\Theta(\eta, \lambda) &= \sqrt[3]{9\lambda} \int_{\chi}^{\infty} \psi d\bar{\chi} = \sqrt[3]{9\lambda} \int_{\chi}^{\infty} \left[\frac{3}{\Gamma(\frac{2}{3})} \int_{\bar{\chi}}^{\infty} \bar{\chi} e^{-\bar{\chi}^3} d\bar{\chi} \right] d\bar{\chi} \\ &= \sqrt[3]{9\lambda} \frac{3}{\Gamma(\frac{2}{3})} \int_{\chi}^{\infty} \bar{\chi} e^{-\bar{\chi}^3} \left[\int_{\chi}^{\bar{\chi}} d\bar{\chi} \right] d\bar{\chi} \\ &= \sqrt[3]{9\lambda} \frac{3}{\Gamma(\frac{2}{3})} \int_{\chi}^{\infty} \bar{\chi} e^{-\bar{\chi}^3} [\bar{\chi} - \chi] d\bar{\chi}\end{aligned}$$

Next rewrite the result as

$$\Theta(\eta, \lambda) = \frac{\sqrt[3]{9\lambda}}{\Gamma(\frac{2}{3})} \left[3 \int_{\chi}^{\infty} \bar{\chi}^2 e^{-\bar{\chi}^3} d\bar{\chi} - 3\chi \left(\int_0^{\infty} \bar{\chi} e^{-\bar{\chi}^3} d\bar{\chi} - \int_0^{\chi} \bar{\chi} e^{-\bar{\chi}^3} d\bar{\chi} \right) \right]$$

The first integration can be performed analytically, the second integral can be put in the form of a (complete) gamma function, and the third integral can be written as an incomplete gamma function. In the second integral, we set $t = \bar{\chi}^3$, so that $dt = 3\bar{\chi}^2 d\bar{\chi}$ and $d\bar{\chi} = \frac{1}{3} t^{-2/3} dt$. Then

$$\int_0^{\infty} \bar{\chi} e^{-\bar{\chi}^3} d\bar{\chi} = \int_0^{\infty} t^{1/3} e^{-t} \left(\frac{1}{3} t^{-2/3} \right) dt = \frac{1}{3} \int_0^{\infty} t^{(2/3)-1} e^{-t} dt = \frac{1}{3} \Gamma\left(\frac{2}{3}\right)$$

A similar derivation can be performed for the indefinite integral. Then we are led directly to Eq. 12.2-24.

12D.8 Forced conduction heat transfer from a flat plate (thermal boundary layer extends beyond the momentum boundary layer)

Evaluate the left side of Eq. 12.4-5 by using Eq. 12.4-8:

$$k \frac{\partial T}{\partial y} \Big|_{y=0} = \frac{2k(T_\infty - T_0)}{\delta_T}$$

Next, evaluate the the integral on the right side of Eq. 12.4-5:

$$\int_0^\infty \rho \hat{C}_p v_x (T_\infty - T) dy = \rho \hat{C}_p v_\infty (T_\infty - T_0) \delta_T \int_0^1 \left(\frac{v_x}{v_\infty} \right) \left(1 - \frac{T - T_0}{T_\infty - T_0} \right) d\eta_T$$

The dimensionless integral on the right side can now be split into two parts to perform the integration, since the equation for the velocity profile changes before we get to the upper limit:

$$\begin{aligned} & \int_0^{1/\Delta} (2\eta_T \Delta - 2\eta_T^3 \Delta^3 + \eta_T^4 \Delta^3) (1 - 2\eta_T + 2\eta_T^3 - \eta_T^4) d\eta_T \\ & + \int_{1/\Delta}^1 (1) (1 - 2\eta_T + 2\eta_T^3 - \eta_T^4) d\eta_T \\ & = \left(\frac{7}{10} \Delta^{-1} - \frac{13}{15} \Delta^{-2} + \frac{67}{140} \Delta^{-4} - \frac{7}{36} \Delta^{-5} \right) + \left(\frac{3}{10} - \Delta^{-1} + \Delta^{-2} - \frac{1}{2} \Delta^{-4} + \frac{1}{5} \Delta^{-5} \right) \\ & = \frac{3}{10} - \frac{3}{10} \Delta^{-1} + \frac{2}{15} \Delta^{-2} - \frac{3}{140} \Delta^{-4} + \frac{1}{180} \Delta^{-5} \equiv F(\Delta) \end{aligned}$$

We can now write Eq. 12.4-5 as

$$\frac{2k(T_\infty - T_0)}{\delta_T} = \frac{d}{dx} \rho \hat{C}_p v_\infty (T_\infty - T_0) \delta_T F(\Delta) \quad \text{or} \quad \frac{2\alpha}{v_\infty} = F(\Delta) \delta_T \frac{d\delta_T}{dx}$$

Integration of this equation then gives $\delta_T = \sqrt{4(\alpha x / v_\infty) / F(\Delta)}$. The result in Eq. 12.4-12 is valid for $\Delta \geq 1$, so that the ratio of the two boundary-layer thicknesses is

$$\Delta = \frac{\delta_T}{\delta} = \sqrt{\frac{4(\alpha x / v_\infty)}{F(\Delta)} \cdot \frac{37}{1260} \left(\frac{v_\infty}{vx} \right)} = \sqrt{\frac{37}{315 \text{Pr} F(\Delta)}}$$

When this is squared, we get Eq. 12D.8-1.